PHYSICS 846

Home Work Assignment # 2 1/20/2009

Due: Tues., Jan. 27, 2009

1. *Thermodynamics:* Show that the coefficient of thermal expansion

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

must vanish in the limit $T \to 0$.

2. *More Thermodynamics:* We want to relate the specific heats at constant volume and pressure, C_V and C_p, to the isothermal and adiabatic compressibilities:

$$\kappa_T = \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_T \quad \text{and} \quad \kappa_S = \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_S$$

and the coefficient of thermal expansion defined in Problem 1.

We will proceed as follows.

(a) To begin with show that

$$Tds = C_V dT + T\left(\frac{\partial p}{\partial T}\right)_V dV$$

and

$$Tds = C_p dT - T\left(\frac{\partial V}{\partial T}\right)_p dp.$$

(b) Next derive the identities

$$\left(\frac{\partial z}{\partial x}\right)_y = 1/\left(\frac{\partial x}{\partial z}\right)_y$$

and

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

and use these to express the second term on the RHS of the results of part (a) in terms of α and κ.

(c) Next, obtain $(C_p - C_V)$ from the the result of part (b). Choose p and V as the independent variables and express dT in terms of changes dp and dV. Thus obtain

$$C_p - C_V = \frac{TV\alpha^2}{\kappa_T}.$$

Note that this shows that $C_p > C_v$ for any substance.

(d) Now consider an adiabatic process and use the results of part (a) to find

$$C_v = -T \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_S$$

and an analogous result for C_p. Find the ratio C_p/C_v.

(e) Using the partial derivative identities of part (b) and the result

$$\left(\frac{\partial x}{\partial y}\right)_w \left(\frac{\partial y}{\partial z}\right)_w = \left(\frac{\partial x}{\partial z}\right)_w,$$

which you should derive, show that

$$\frac{C_p}{C_V} = \frac{\kappa_T}{\kappa_S}.$$

It thus follows that $\kappa_T > \kappa_S$.

(f) Using the results for $(C_p - C_V)$ and C_p/C_V derived above determine C_p and C_v.

(g) Check the general relations derived above for the special case of the ideal gas.

3. *Some useful mathematical tricks:*
(a) Show that the volume of a sphere of radius R in D-dimensions is given by the result:

$$\int dx_1 dx_2 \ldots dx_D \Theta \left(R^2 - \sum_{i=1}^{D} x_i^2 \right) = \frac{\pi^{D/2}}{\Gamma(D/2+1)} R^D$$

Here $\Theta(x) = 1$ for $x \geq 0$, and $= 0$ for $x < 0$. Check that this formula reduces to known answers in $D = 1, 2, 3$.

(b) Show that in spaces of high dimensionality essentially all the volume of a ball is contained in a thin shell around the surface.

This counterintuitive result is actually quite simple. Consider the fraction of the volume contained within a smaller sphere of radius xR, with $x < 1$, inside the ball of radius R and take the limit of the dimensionality $D \to \infty$.

4. *Statistical Mechanics of a Classical Ideal Gas in a Harmonic Trap:*
A spherically symmetric harmonic trap, used to confine cold atoms, exerts a radially directed linear restoring force $\mathbf{F} = -m\omega^2 \mathbf{r}$ on an atom displaced by $\mathbf{r}$ from the trap center. Here ω is the trap frequency and m the atomic mass.

Consider N atoms with a total energy E placed in such a trap under conditions when they can be treated as a *classical, ideal* gas. We will thus ignore interatomic interactions, and also ignore quantum (Bose/Fermi) statistics, and treat the atoms as classical, indistinguishable particles.

(a) Write down the classical Hamiltonian H of the system in terms of the $6N$ variables $\mathbf{p}_i$ and $\mathbf{r}_i$ for $i = 1, \ldots, N$.

(b) Show that in the **microcanonical ensemble** the number of number of microstates accessible to the system is given by

$$\Omega(E, N) = \frac{1}{N!} \int_{E-\text{shell}} \prod_{i=1}^{N} \left(\frac{d^3\mathbf{p}_i d^3\mathbf{r}_i}{h^3} \right)$$

where the integral is over a thin shell around the specified energy E. Explain the origin of each of the factors in the expression above.

(c) Rescale variables so that Ω is now expressed (in the new variables) as an integral over the surface of a spherical shell of radius unity in $6N$ dimensions.

(d) Using the result of Problem 3(b) show that, in the limit of large N, we can extend the integral to the entire volume of a sphere of radius unity in $6N$ dimensions, without significantly altering thermodynamical predictions.

(e) Next use the result of Problem 3(a) and determine the *entropy* of the trapped atomic gas. Assume that the number of particles is large $N \gg 1$ and work to leading order in this large number. Is the entropy extensive? If not, why not?

(f) Find the *internal energy* of the system as a function of temperature T.

(g) Calculate the probability of finding a particle about the point $\mathbf{r}$ at a temperature T using the following procedure.

Consider one of the particles to have a position $\mathbf{r}$ and momentum $\mathbf{p}$. Denote by E_{other} as the energy of the rest of the system, given by E minus the energy of the particle singled out. The probability of finding the particle in this state is then the ratio of microstates with the particle in this particular state, versus it being in an arbitrary state, i.e., $\Omega(E_{\text{other}}, N-1)/\Omega(E, N)$. Use your earlier result for Ω, and expand the result to lowest order in the energy of the particle singled out. Finally integrate over all momenta $\mathbf{p}$ to show that the probability of finding a particle at $\mathbf{r}$ is a Gaussian. What is the normalization of this gaussian?

(h) Find the mean radius $\langle r \rangle$ of the atomic cloud at temperature T. Estimate this radius for a trap frequency $\nu = \omega/2\pi = 100$ Hz for sodium atoms (atomic weight 23) at room temperature.