PHYSICS 880A20

Assignment # 3 10/17/2007

H.W. due October 24, 2007

A brief outline of your Term Paper is due by Monday, October 29. This should include: the title of your term paper, a very brief summary of the main points you plan to work on (either in bullet form or in a few sentences), and the main reference(s) that you plan to use.

1. Prove the **f-sum rule**

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega S(\mathbf{q}, \omega) = \frac{nq^2}{2m}$$

where S is the dynamic structure factor for density fluctuations, n is the total density and m the mass of the particles. There is a fair bit of algebra involved and I suggest that you follow these steps:

(a) First, prove the following identity (which is closely related to the continuity equation)

$$[\rho_{\mathbf{q}}, \mathcal{H}] = \mathbf{q} \cdot \mathbf{J}_{\mathbf{q}}$$

where ρ is the density operator, $\mathbf{J}$ the current operator, and $\mathcal{H} = T + V$ the Hamiltonian. Hint: It is trivial to see that $[\rho_{\mathbf{q}}, V] = 0$ in the "first quantized" representation, and you only need to evaluate the commutator between $\rho_{\mathbf{q}}$ and T. I suggest using "second quantization" here and in part (b), but you can do it any way you want to.

(b) Second, explicitly evaluate the commutator below to show that

$$\mathbf{q} \cdot [\mathbf{J}_{\mathbf{q}}, \rho_{-\mathbf{q}}] = \frac{Nq^2}{m}.$$

(c) Next, use exact eigenstates $\mathcal{H}|\psi_a\rangle = \omega_a|\psi_a\rangle$ to show that the thermal expectation value of the double commutator on the left-hand side has the spectral representation

$$\langle [[\rho_{\mathbf{q}}, \mathcal{H}], \rho_{-\mathbf{q}}] \rangle = 2 \sum_{a,b} \frac{e^{-\beta\omega_a}}{\mathcal{Z}} \omega_{b,a} \left| (\rho_{\mathbf{q}})_{a,b} \right|^2$$

where $\omega_{b,a} = \omega_b - \omega_a$ and $(\rho_{\mathbf{q}})_{a,b} = \langle \psi_a | \rho_{\mathbf{q}} | \psi_b \rangle$.

(d) Finally put together the results of parts (a), (b) and (c) and the definition of $S(\mathbf{q}, \omega)$ to obtain the stated form of the f-sum rule.

2. **Single mode approximation (SMA)**: Here we will derive results closely related to Feynman's variational approach for Helium-4 (discussed in class) using an approach based entirely on sum-rules.

We start with the ansatz that a single mode with dispersion $\omega = \omega(\mathbf{q})$ dominates the spectrum of density fluctuations, so that the $T = 0$ dynamic structure factor is given by

$$S(\mathbf{q}, \omega) = F(\mathbf{q}) \delta \left(\omega - \omega(\mathbf{q}) \right)$$

where both $\omega(\mathbf{q})$ and $F(\mathbf{q})$ are unknown and will be constrained by sum rules.

(a) First use the relation between $S(\mathbf{q}, \omega)$ and the static structure factor $S(\mathbf{q})$ to express F in terms of the latter.

(b) Next use the f-sum rule to determine the spectrum $\omega(\mathbf{q})$ in terms of $S(\mathbf{q})$.

(c) Finally use the compressibility sum rule to determine the long-wavelength forms of $\omega(\mathbf{q})$ and $S(\mathbf{q})$.

Note that the SMA is essentially exact for the dilute Bose gas whose $S(\mathbf{q}, \omega)$ was derived in class using Bogoliubov theory. However, it is of much more general validity as a useful approximation for strongly interacting Bose systems.

3. **Reading Assignment:** Review the ground state and low temperature ($T \ll E_f$) properties of an ideal **Fermi gas** from your favorite Statistical Mechanics or Solid State Physics book. In particular, review the following concepts: Fermi surface, Fermi energy, density of states and its relation to low temperature properties like the linear T specific heat.