PHYSICS 880A20

Assignment # 2 10/3/2007

H.W. due October 12, 2007

Please note:
(1) Extra class: Friday, October 5 at 2:30 PM
in Room AV 115 (our regular classroom)
(2) No class on Wednesday, October 10
(3) HW should be placed in the Mr. Qi Zhou's mail box.

1. Let $|\Psi_0\rangle$ be the normalized, N-particle ground state of an interacting many body system of bosons or of (spinless) fermions. Using the $T = 0$ (many-body) density matrix $\widehat{\rho} = |\Psi_0\rangle\langle\Psi_0|$ we define the one-particle, reduced density matrix $\widehat{\rho}_1$ by

$$\langle \mathbf{r}'|\widehat{\rho}_1|\mathbf{r}\rangle = N \int d\mathbf{r}_2 \ldots d\mathbf{r}_N \langle \mathbf{r}', \mathbf{r}_2 \ldots \mathbf{r}_N|\widehat{\rho}|\mathbf{r}, \mathbf{r}_2 \ldots \mathbf{r}_N\rangle$$

Here $|\mathbf{r}_1 \ldots \mathbf{r}_N\rangle = (N!)^{-1/2}\, \psi^\dagger(\mathbf{r}_N) \ldots \psi^\dagger(\mathbf{r}_1)|0\rangle$ (note order) are a complete set of states in the N-particle Hilbert space.

(a) Show that it trivially follows from the definition that

$$\langle \mathbf{r}'|\widehat{\rho}_1|\mathbf{r}\rangle = N \int d\mathbf{r}_2 \ldots d\mathbf{r}_N \Psi_0(\mathbf{r}', \mathbf{r}_2 \ldots \mathbf{r}_N)\Psi_0^*(\mathbf{r}, \mathbf{r}_2 \ldots \mathbf{r}_N)$$

(b) Show that $\mathrm{Tr}\widehat{\rho}_1 = N$.

(c) Show that $\langle \mathbf{r}'|\widehat{\rho}_1|\mathbf{r}\rangle = \langle\Psi_0|\psi^\dagger(\mathbf{r})\psi(\mathbf{r}')|\Psi_0\rangle$

(d) Show that for a translationally invariant system $\rho_1(\mathbf{r}' - \mathbf{r}) = \langle \mathbf{r}'|\widehat{\rho}_1|\mathbf{r}\rangle$ is the Fourier transform of the momentum distribution $n(\mathbf{k}) = \langle\Psi_0|a^\dagger_{\mathbf{k}} a_{\mathbf{k}}|\Psi_0\rangle$.

(e) Using the result of part (d) compute $\rho_1(\mathbf{r} - \mathbf{r}')$ for
  (i) an ideal Bose gas at $T = 0$
  (ii) an ideal Fermi gas at $T = 0$

(f) Note the characteristic singularities in $n(\mathbf{k})$ for the two cases and discuss how they influence the long distance behavior of $\rho_1(\mathbf{r} - \mathbf{r}')$.

(g) Try to see how much more cumbersome it is to derive the result of part (e) in the Fermi case directly from the "first quantized" many-body wavefunction result of part (a), instead of using (d). Start by writing down the many-body ground state wavefunction $\Psi_0(\mathbf{r}_1 \ldots \mathbf{r}_N)$ for the ideal Fermi gas and indicate how you would proceed, without necessarily carrying out the full calculation.

2. Consider the *classical* damped, harmonic oscillator whose equation of motion is given by

$$\ddot{x} + \gamma \dot{x} + \Omega^2 x = f(t)/m$$

Here $x(t)$ is the displacement, m the mass of the particle, Ω the natural frequency in the absence of damping, γ the damping coefficient and $f(t)$ the external force. Use the F.T. convention

$$x(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} x(\omega)$$

(a) Find the complex susceptibity $\chi(\omega)$ which describes the linear response x of the system to the external force f, i.e., $x(\omega) = \chi(\omega) f(\omega)$.
(b) Where in the complex-z plane are the poles of $\chi(z)$ located? What is their physical meaning?
(c) Calculate $\mathrm{Re}\chi(\omega)$ and $\mathrm{Im}\chi(\omega)$ and plot these functions for $\gamma \ll \Omega$ after determining the even/odd properties of these function under $\omega \to -\omega$.
(d) Show that for a forcing function $f(t) = F_0 \cos(\omega t)$, the steady-state response of the system is of the form $x(t) = F_0 A(\omega) \cos(\omega t - \phi(\omega))$ where $A(\omega) \geq 0$. Find the amplitude $A(\omega)$ and the phase $\phi(\omega)$. How are they related to $\chi(\omega)$?
(e) The rate at which the external force does work on the system is given by $dW/dt = f(t)\dot{x}(t)$. Consider the periodic external force of part (d). In the steady state the average power dissipated can be determined by time-averaging dW/dt over a cycle with period $2\pi/\omega$. Calculate the average power dissipated in the steady state and show that

$$\omega \mathrm{Im}\chi(\omega) \geq 0.$$

(f) Determine the real time response function $\chi(t)$ by calculating the F.T. of $\chi(\omega)$ using contour integration. You must carefully justify the choice of contours in the complex plane by making sure that the integrand vanishes for large $|z|$ so as to ensure convergence.
Comment how this calculation shows that causality (in time) is closely tied to analyticity in the upper half plane (of the complex-ω plane).

3. This problem will take you through the details of the derivation of the Kubo formula for $T \neq 0$ (which were skipped in class). The idea is very similar to that used in class for the $T = 0$ derivation, except now one needs to find the full density matrix to first order in the perturbation (in place of just the ground state wave-function).

Start with the equation of motion for the density matrix ρ given by

$$i\frac{d}{dt}\rho = [\mathcal{H}_{\text{total}}, \rho]$$

where $\mathcal{H}_{\text{total}} = \mathcal{H} + \mathcal{H}'$ where $\mathcal{H}$ is the Hamiltonian of the interacting many-body system and $\mathcal{H}' = -AF_A(t)$ is the coupling of the external perturbation $F_A(t)$ to the the system through the operator A. The initial condition is given by the fact that the system started out in thermal equilibrium at a temperature $T = \beta^{-1}$ in the distant past:

$$\rho(t \to -\infty) = \exp(-\beta\mathcal{H})/\mathcal{Z} \equiv \rho_0, \qquad \mathcal{Z} = \mathrm{Tr}\left\{\exp(-\beta\mathcal{H})\right\}.$$

(a) Switch to the interaction representation in which all operators are transformed according to $\widetilde{Q}(t) = \exp(i\mathcal{H}t)Q\exp(-i\mathcal{H}t)$. Show that the interaction representation equation of motion is given by

$$i\frac{d}{dt}\widetilde{\rho}(t) = \left[\widetilde{\mathcal{H}}'(t), \widetilde{\rho}(t)\right]$$

(b) Solve the equation of motion of part (a) subject to the given initial condition and show that, to first order in the perturbation, the result is

$$\widetilde{\rho}(t) = \rho_0 + i\int_{-\infty}^{t} dt' \left[\widetilde{\rho}_0, \widetilde{\mathcal{H}}'(t')\right] + \ldots$$

(c) We finally want to calculate the response $\langle B\rangle(t) \equiv \mathrm{Tr}\left\{\rho(t)B\right\}$.

First, show that this may be written in the interaction representation as $\langle B\rangle(t) = \mathrm{Tr}\left\{\widetilde{\rho}(t)\widetilde{B}(t)\right\}$. Hint: use the cyclic property of the trace $\mathrm{Tr}\left\{ABC\right\} = \mathrm{Tr}\left\{BCA\right\}$.

(d) Now assume (for simplicity) that $\mathrm{Tr}\left\{\rho_0 B\right\} = 0$ and combine the results obtained above to conclude that

$$\langle B\rangle(t) = \int_{-\infty}^{+\infty} dt'\, \chi(t-t')\, F_a(t')$$

with

$$\chi(t-t') = i\Theta(t-t')\left\langle\left[\widetilde{B}(t), \widetilde{A}(t')\right]\right\rangle_T$$

where $\langle\ldots\rangle_T$ represents an average with respect to the *equilibrium* density matrix ρ_0 at temperature T.