PHYSICS 880A20

Assignment # 1 9/24/2007

Reminder: Please finalize Topic of Term paper by first week of October, preferably Oct. 1

Home Work: due Wednesday, Oct. 3, 2007

1. Estimate the order of magnitude of the degeneracy temperature in Kelvin or eV or any other unit which seems most natural for the following systems:
(a) Dilute alkali gas of ^{87}Rb atoms in BEC experiments;
(b) liquid Helium 4;
(c) electron gas in metals;
(d) neutrons in a neutron star.

2. I use the following Fourier transform (F.T.) conventions for a system with volume Ω. For the creation and annihilation operators

$$a_{\mathbf{k}} = \frac{1}{\sqrt{\Omega}} \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \psi(\mathbf{r}); \qquad \psi(\mathbf{r}) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} e^{+i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}}$$

These can be either bosons or fermions (with the spin index omitted for simplicity). For all other operators

$$B_{\mathbf{q}} = \int d\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} B(\mathbf{r}); \qquad B(\mathbf{r}) = \frac{1}{\Omega} \sum_{\mathbf{q}} e^{+i\mathbf{q}\cdot\mathbf{r}} B_{\mathbf{q}}.$$

In addition: $\int d\mathbf{r} e^{+i\mathbf{q}\cdot\mathbf{r}} = \Omega \delta_{\mathbf{q},0}$.
(a) The "first quantized" form of the current operator is given by

$$\mathbf{J}(\mathbf{r}) = \frac{1}{2m} \sum_{i=1}^{N} \left[\widehat{\mathbf{p}}_i \delta(\mathbf{r} - \widehat{\mathbf{r}}_i) + \delta(\mathbf{r} - \widehat{\mathbf{r}}_i) \widehat{\mathbf{p}}_i \right]$$

Using the basis set $|\mathbf{r}_1 \dots \mathbf{r}_N\rangle = (N!)^{-1/2}\, \psi^\dagger(\mathbf{r}_1) \dots \psi^\dagger(\mathbf{r}_N)|0\rangle$, or otherwise, show that the "second quantized" form is given by

$$\mathbf{J}(\mathbf{r}) = \frac{-i}{2m} \left[\psi^\dagger(\mathbf{r}) \nabla \psi(\mathbf{r}) - \left(\nabla \psi^\dagger(\mathbf{r}) \right) \psi(\mathbf{r}) \right].$$

(b) Show that the F.T. of the above result is given by

$$\mathbf{J_q} = \sum_{\mathbf{k}} (\mathbf{k}/m) a^\dagger_{\mathbf{k}-\mathbf{q}/2} a_{\mathbf{k}+\mathbf{q}/2}.$$

(c) Show that the kinetic energy operator $T = \sum_{\mathbf{k}} (k^2/2m) a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$ can be written as

$$T = \frac{-1}{2m} \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \nabla^2 \psi(\mathbf{r})$$

(d) Show that the potential energy operator

$$V = \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' V(\mathbf{r} - \mathbf{r}') \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r})$$

can be written as

$$V = \frac{1}{2\Omega} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_{\mathbf{q}} a^\dagger_{\mathbf{k}+\mathbf{q}/2} a^\dagger_{\mathbf{k}'-\mathbf{q}/2} a_{\mathbf{k}'+\mathbf{q}/2} a_{\mathbf{k}-\mathbf{q}/2}$$

3. Consider boson *coherent states* defined by

$$|z\rangle = \exp\left(-|z|^2/2\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

where z is any complex number, and $|n\rangle$ is the Fock state with n bosons.
(a) Show that $|z\rangle$ is a normalized (right) eigenstate of the boson annihilation operator a and find the corresponding eigenvalue.
(b) Can you find a (right) eigenstate of the creation operator $a^\dagger$? If so, do it. If not, explain why.
(c) Show that the set of all coherent states $\{|z\rangle\}$ is complete basis set by proving that

$$\frac{1}{\pi} \int d^2 z |z\rangle\langle z| = \widehat{\mathbf{1}}.$$

(d) Show that the set of all coherent states is <u>over</u>complete.
(e) A coherent state is a linear superposition of states with different numbers of particles. Find the mean number of bosons and the fluctuations from the mean by calculating $\langle z|N|z\rangle$ and $\langle z|N^2|z\rangle$.

Later on in the course you will see other examples of states which are linear superpositions of states with different numbers of particles (the BCS ground state, e.g.). In the second quarter you will also make use of coherent path integrals.