PHYSICS 829

Home Work Assignment # 5 5/9/2011

Due: Mon., May 16, 2011.

Review of Classical Electrodynamics:

1. Continuity equation: Shankar Ex. 18.4.1 (p. 492)

2. Gauge invariance: Shankar Ex. 18.4.2 (p. 493)

3. Coulomb (or transverse) gauge: Shankar Ex. 18.4.3 (p. 494)

Quantum particle interacting with classical electromagnetic fields:

4. Gauge invariance in QM: Shankar Ex. 18.4.4 (p. 497)

5. Continuity equation in QM: Define a gauge-invariant current and check the continuity equation for the probability density $\rho(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2$.

6. Aharanov-Bohm Effect: Consider a charged particle, of charge q and mass M, confined to the interior of a "toroidal" region:

$$a < r < b, \quad 0 \leq \varphi < 2\pi, \quad -h < z < +h,$$

where we use a cylindrical coordinate system. A magnetic flux Φ threads the "hole" of the torus as a result of a long thin solenoid (radius $\ll a$) oriented along the z-axis. (The scalar potential is identically zero, so we can use the symbol Φ for the magnetic flux without any cause for confusion).

(a) Show that the vector potential

$$A_\varphi = \frac{\Phi}{2\pi r}, \qquad A_r = A_z = 0$$

can be used to describe the magnetic field.

(b) Write down the time-independent Schrödinger equation for the charged particle, and show that the energy eigenfunctions are of the form

$$\Psi(r, \varphi, z) = R_n(r) e^{im\varphi} \sin\left[\frac{j\pi(z+h)}{2h}\right].$$

For the given boundary conditions, what are the allowed values of j and m?

(c) Show that the radial wavefunction $R_n(r)$ is a solution of the equation

$$-\frac{\hbar^2}{2M}\left[\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \frac{(m - \Phi/\Phi_0)^2}{r^2} R_n - k_z^2 R_n\right] = E R_n$$

where $k_z = j\pi/2h$ and $\Phi_0 = hc/e$ is the flux quantum, and $R_n(r)$ satisfies the boundary conditions $R_n(a) = R_n(b) = 0$. This equation can be solved in terms of Bessel functions but that will not be necessary for our purposes.

(d) Using the result of part (c) argue that the energy E of a stationary state must depend on the magnetic flux Φ even though the electron only moves in the region $a < r < b$ where the magnetic field is zero, and the flux exists only in the region $r \ll a$.

7. Absorption and emission of radiation:

Consider an atom in an external electric field. In the dipole approximation, we may write the perturbation as

$$\mathcal{H}_1(t) = e\mathbf{R}\cdot\mathbf{E_0}\left(e^{-i\omega t} + e^{+i\omega t}\right)$$

for $0 < t < T$ and $\mathcal{H}_1(t) = 0$ at other times. We use here $(-e)$ as the charge of the electron.

(a) Let the initial state of the system be an eigenstate $|i\rangle$ of the atomic Hamiltonian $\mathcal{H}_0$ with energy E_i. Show using standard time-dependent perturbation theory that the probability that the atom will be in a state $|f\rangle$ of energy E_f at any time $t \geq T$ is given by $|a_f(T)|^2$ where

$$a_f(T) = \frac{\langle f|H_1|i\rangle}{\hbar}\left[\frac{1 - e^{i(\omega_{fi}-\omega)T}}{\omega_{fi} - \omega}\right] + \frac{\langle f|H_1^\dagger|i\rangle}{\hbar}\left[\frac{1 - e^{i(\omega_{fi}+\omega)T}}{\omega_{fi} + \omega}\right]$$

where $\hbar\omega_{fi} = (E_f - E_i)$ and $\mathcal{H}_1(t) = H_1 e^{-i\omega t} + H_1^\dagger e^{+i\omega t}$.

(b) For near resonant absorption with $\omega \approx \omega_{fi}$, show that

$$|a_f(T)|^2 \simeq \frac{e^2}{\hbar^2} |\langle f|\mathbf{R} \cdot \mathbf{E}_0|i\rangle|^2 \left[\frac{\sin[(\omega_{fi} - \omega)T/2]}{(\omega_{fi} - \omega)/2}\right]^2 \qquad (1)$$

(c) In the limit of large T, show that the factor $[\ldots]^2$ in eq. (1) can be replaced by $2\pi T \delta(\omega - \omega_{fi})$.

We next want to consider absorption from "incoherent" radiation with a continuous frequency spectrum, for which we can integrate the probability in eq. (1) above over the incident spectrum $u(\omega)$ (defined below). (Otherwise we'd have to first integrate the amplitude over the spectrum and then find the probability). To get all the prefactors right, we argue as follows. You do not need to do the algebra in the steps below where it says "one can show" as those steps have little to do with quantum mechanics per se.

(i) For unpolarized radiation, averaging over the directions of $\mathbf{E}_0$ one can show that $|\langle f|\mathbf{R} \cdot \mathbf{E_0}|i\rangle|^2 \to \frac{1}{3}|\mathbf{E_0}|^2 |\langle f|\mathbf{R}|i\rangle|^2$

(ii) One can further show that $|\mathbf{E_0}|^2$ can be replaced by $2\pi u(\omega)$, where $u(\omega)$ is the time-averaged energy density per unit ω. This follows from using the expression $|\mathbf{E_0}|^2/4\pi$ as the energy density in the radiation field including both E and B fields, noting the form $\mathbf{E} = 2\mathbf{E_0}\cos(\omega t)$ used in part (a) and time-averaging.

(d) Put together the results of part (c) with (i) and (ii) above to obtain the *rate of absorption* of energy

$$R_{\text{abs}} = \frac{1}{T}\int |a_f(T)|^2 d\omega = \frac{4\pi^2}{3}\frac{e^2}{\hbar^2} u(\omega_{fi}) |\langle f|\mathbf{R}|i\rangle|^2.$$

One can show that an almost identical calculation – focusing on the second term in (a) – gives the *same* result for rate of *stimulated emission* of radiation. Here stimulated emission is the de-excitation from a (higher energy) initial state at energy E_i to a (lower energy) final state at energy E_f in the presence of a field oscillating at frequency $\omega \approx (E_i - E_f)/\hbar = -\omega_{fi}$.

8. Spontaneous emission and Einstein's A and B coefficients:

Experimentally it is well known that an atom in an excited state will spontaneously emit radiation and make a transition to a lower energy state. A microscopic derivation of *spontaneous emission* goes beyond a classical treatment of the EM field and requires quantizing it. Here we present a simple argument due to Einstein that permits one to relate the rate of spontaneous emission to that for absorption using simple statistical ideas. In fact, this argument (1917) predates the Schrödinger equation (1926)!
This problem assumes some knowledge of elementary statistical mechanics. Let us write the result of Prob. 7 for the rate of *absorption* as $R_{\text{abs}} = B_{if}\, u(\omega_{fi})$. The rate of *stimulated emission* is similarly $B_{fi}\, u(\omega_{fi})$, where the *principle of detailed balance* implies $B_{fi} = B_{if}$. The quantum calculation in Prob. 7 explicitly showed this equality.

(a) Let $N(i)$ be the number of atoms in the i^{th} state. In equilibrium the rates of transition from $i \to f$ must balance those from $f \to i$. Show that this leads to the condition

$$B_{fi}\, u(\omega_{fi}) N(f) + A_{fi} N(f) = B_{if}\, u(\omega_{fi}) N(i),$$

where the second term on the LHS represents *spontaneous emission* that exists independent of the presence of any radiation.

(b) Using the Boltzmann distribution

$$N(i)/N(f) = \exp(-E_i/k_B T)/\exp(-E_f/k_B T)$$

for atoms in thermal equilibrium, show that the result of part (a) leads to

$$u(\omega_{fi}) = \frac{A_{fi}/B_{fi}}{[\exp(\hbar\omega_{fi}/k_B T) - 1]}.$$

(c) Now we can use Planck's result for black-body radiation to determine the A/B ratio. Even more simply, we can argue that A and B are microscopic quantum mechanical probabilities that must be independent of temperature. Thus we take the high-temperature, low-frequency limit of the result of part (b) and equate it to the classical Rayleigh-Jeans formula $u(\omega_{fi}) \approx (\omega_{fi}^2/\pi^2 c^3) k_B T$ to obtain

$$A_{fi} = \frac{\hbar\omega_{fi}^3}{\pi^2 c^3} B_{fi}.$$

Given that we already calculated B_{fi} in Problem 7, this result tells us what A_{fi} must be.