PHYSICS 829

Home Work Assignment # 3 4/15/2011

Note: New due date: Mon., Apr. 25, 2011.
Please see "new" problems 6, 7 and 8 on page 3.

1. Shankar Ex. 19.3.3 (p. 533)

2. Shankar Ex. 19.5.1 (p. 546)

3. As shown in class, the asymptotic form of the wavefunction in a scattering experiment is given by

$$\psi_k(\mathbf{r}) \sim e^{ikz} + \frac{1}{2ik}\left[\sum_{\ell=0}^{\infty}(2\ell+1)\,(S_\ell(k)-1)\,P_\ell(\theta)\right]\frac{e^{ikr}}{r}$$

as $r \to \infty$. Here $E = \hbar^2 k^2/2m$ is the energy and $S_\ell(k) = e^{i2\delta_\ell(k)}$ is the "S-matrix" for angular momentum ℓ.

(a) Show that the scattering amplitude $f(\theta) = \sum_\ell (2\ell+1) f_\ell(\theta) P_\ell(\theta)$ with the following two equivalent forms for f_ℓ which are useful: $f_\ell = e^{i\delta_\ell}\sin\delta_\ell/k$ and $f_\ell = 1/[k\cot\delta_\ell - ik]$.

(b) Show that the *total* scattering cross-section can be written as $\sigma = \sum_\ell \sigma_\ell$, with $\sigma_\ell = 4\pi(2\ell+1)\sin^2\delta_\ell/k^2$. What is the maximum σ_ℓ for a given energy? This is called the unitarity bound.

(c) Prove the *optical theorem* $\sigma = 4\pi \mathrm{Im} f(0)/k$, relating the total cross section to the forward scattering amplitude.

4. Consider a partial wave analysis of scattering off a spherically symmetric potential $V(r)$, which is non-zero for $r < R$ and vanishes identically for $r > R$. Let $R_{k,\ell}(r)$ be the solution of the radial wavefunction, for $r \leq R$, corresponding to an energy $E = \hbar^2 k^2/2m$. Show that the phase shift is given by the solution of

$$\cot\delta_\ell(k) = \frac{k\,n'_\ell(kR) - \alpha_\ell\,n_\ell(kR)}{k\,j'_\ell(kR) - \alpha_\ell\,j_\ell(kR)}$$

where $\alpha_\ell = d\ln R_{k,\ell}/dr$ evaluated at $r = R$. We use the usual symbols for the spherical Bessel and Neumann functions and primes denote derivatives.

5. Scattering length and bound states

Consider the square well potential of depth $V_0\,(>0)$ and range r_0, given by $V(r) = -V_0\Theta(r_0 - r)$. We want to compute the s-wave scattering length and cross-section, but to understand the behavior of these quantities, we begin by considering possible bound states in this potential. We will focus only on the $\ell = 0$ angular momentum solutions and their properties.

(a) Define $\hbar^2 k_0^2/2m = V_0$. Show that, as the well depth is increased, a new $\ell = 0$ bound state with $E < 0$ is obtained each time $k_0 r_0$ crosses $(2n+1)\pi/2$ where $n = 0, 1, 2, \ldots$. (You more-or-less solved this problem last quarter).

(b) Now solve the problem for $E = \hbar^2 k^2/2m > 0$ scattering states and show that the s-wave phase shift $\delta_0(k)$ must satisfy

$$\cot(kr_0 + \delta_0) = \frac{K}{k}\cot(Kr_0),$$

where $K^2 = k_0^2 + k^2$.

(c) Let us confine ourselves to the low energy regime $kr_0 \ll 1$.
We will focus on the vicinity of the threshold for the formation of the first bound state: $k_0 r_0 = \pi/2 + \epsilon$ with $|\epsilon| \ll 1$.
Working to lowest order in kr_0 and in ϵ, simplify the result of part (b) to show that $\cot\delta_0 \approx -C\epsilon/(kr_0)$, where C is a positive constant that you can find.

(d) The *s-wave scattering length* a_s is defined by the $k\cot\delta_0(k) \approx -1/a_s$ in the $kr_0 \to 0$ limit. Show that the scattering length diverges to $\pm\infty$ as the potential well depth is tuned across unitarity, the threshold for two-body bound state formation.

(e) Sketch a_s/r_0 as a function of $k_0 r_0$, a dimensionless measure of the well-depth. Also sketch the phase shift $\delta(k \to 0)$ as a function of $k_0 r_0$.

(f) In the regime just beyond threshold, where $a_s > 0$, how is the scattering length related to the size of the bound state, and to its binding energy?

(g) Using the results of Problem 3, find the total scattering cross section and plot it as a function of energy or k, for a large but finite a_s. Note the resonance.

Additional Problems on Time-dependent Hamiltonians:

6. Shankar Ex. 18.2.2 (p. 476)

7. Read Shankar p. 477 on the "Sudden approximation" and then do Ex. 18.2.3 (p. 478).

8. Shankar Ex. 18.2.6 (p. 478).