PHYSICS 828

Home Work Assignment # 6 2/25/2011

Due: Fri., Mar. 4, 2011.

Completed assignments should be placed in the grader N. Ramalingam's mail box in PRB by 5:00 PM.

1. Recall the *Heisenberg picture* for a system described by a time-independent Hamiltonian $\mathcal{H}$. In this representation the states have no time evolution, so that $|\psi'\rangle = |\psi(t=0)\rangle$, and operators acquire a time dependence via

$$\mathcal{A}'(t) = \exp(+i\mathcal{H}t/\hbar)\ \mathcal{A}\ \exp(-i\mathcal{H}t/\hbar).$$

Note that I use the "prime" notation to denote Heisenberg picture states and operators, and those without primes are in the usual *Schrödinger picture.*
(a) Show the equivalence of the two pictures by showing that

$$\langle\chi(t)|\mathcal{A}|\psi(t)\rangle = \langle\chi'|\mathcal{A}'(t)|\psi'\rangle$$

(b) Show that the Heisenberg equation of motion (that replaces the Schrödinger equation) is

$$\frac{d}{dt}\mathcal{A}'(t) = \frac{i}{\hbar}\left[\mathcal{H}, \mathcal{A}'(t)\right],$$

where we assume that $\mathcal{A}$ has no explicit time-dependence.

Now consider the problem of *spin precession in an external magnetic field* in the Heisenberg picture. (In class we used the Schrödinger picture).
(c) Consider a spin $\mathbf{S}$ in an external (time-independent) field $\mathbf{B}$. Show that:

$$\frac{d}{dt}\mathbf{S}' = \mathbf{S}' \times \gamma\mathbf{B}.$$

You only need to explicitly compute, e.g., dS'_x/dt and argue by symmetry. Note that this operator equation has the same structure as the classical equation of motion!
(d) Specialize to $\mathbf{B} = B_0\hat{z}$ and solve the equations of motion to for $S'_\alpha(t)$ (where $\alpha = x, y, z$) given $S'_\alpha(0)$. Show that this leads to same precession that we had deduced from our Schrödinger picture analysis.

2. Shankar Ex. 14.5.3 (p. 401).

3. Consider two spinless particles of mass M_1 and M_2 moving in a central potential $V_0(r)$ and interacting with each other via $V(r_{12})$ where $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$. The Hamiltonian is

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + V(r_{12}) \quad \text{with} \quad \mathcal{H}_i = -\frac{\hbar^2}{2M_i}\nabla_i^2 + V_0(r_i).$$

Show that $\mathbf{L}_1$ and $\mathbf{L}_2$ are not conserved. However, the total orbital angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ is, i.e.,

$$[\mathbf{L}, \mathcal{H}] = 0.$$

It is sufficient for you to show that this is true for L_z since very similar algebra works for the other components (or you can appeal to rotational invariance).

4. Consider spin-orbit interaction described by the Hamiltonian

$$\mathcal{H}_{\text{so}} = \lambda_{\text{so}}\ \mathbf{L} \cdot \mathbf{S}.$$

(a) Show that although neither $\mathbf{L}$ nor $\mathbf{S}$ are conserved, we have

$$[\mathbf{J}, \mathcal{H}_{\text{so}}] = 0,$$

where $\mathbf{J} = \mathbf{L}+\mathbf{S}$ is the total angular momentum. As in the previous problem, it is sufficient to show these assertions for the z-components of the angular momenta.
(b) Show that $\left[L^2, \mathcal{H}_{\text{so}}\right] = 0$ and $\left[S^2, \mathcal{H}_{\text{so}}\right] = 0$.
(c) Thus it follows from (a) and (b) that L^2, S^2, J^2,and J_z form a set of commuting observables. Find the eigenvalues of $\mathcal{H}_{\text{so}}$.

5. Shankar 15.1.1 (page 405). Please also derive eq. (15.1.12).

6. Shankar 15.1.2 (page 407 - 408).

7. Shankar 15.2.2 (page 413). Part (2) will be done in class.

8. Shankar 15.2.5 (page 415).

9. Consider two spin-half particles interacting via the Hamiltonian

$$\mathcal{H} = \alpha\ \mathbf{S}_1 \cdot \mathbf{S}_2$$

where the coupling constant is $\alpha > 0$.
Find the eigenstates and eigenvalues, noting their degeneracies.