PHYSICS 828

Home Work Assignment # 2 1/14/2011

Due: Fri., Jan. 21, 2011.

Completed assignments should be placed in the grader N. Ramalingam's mail box in PRB by 5:00 PM.

1. Show that if the Hamiltonian of a system is invariant under space inversion *and* if the state of the system is non-degenerate, then there can be *no spontaneous electric dipole moment* in that state.

Lets break up the proof into several parts. First, lets figure out the consequences of spatial inversion Π for a non-degenerate energy eigenstate.
(1) Show that if $|\psi\rangle$ is an eigenstate with energy E, then so is $|\psi'\rangle = \Pi|\psi\rangle$.
(2) Then, using non-degeneracy, show that $|\psi'\rangle = c|\psi\rangle$, where $c = \pm 1$.

Next, consider the electric dipole *operator* for a multi-particle system:

$$\mathbf{D} = \sum_j q_j \mathbf{R}_j$$

where q_j is the charge and $\mathbf{R}_j$ the position *operator* of the j-th particle. (Here each operator $\mathbf{R}_j$ is a three dimensional vector).
(3) Show that the electric dipole operator has odd parity:

$$\Pi^\dagger \, \mathbf{D} \, \Pi = -\mathbf{D}.$$

A stationary state $|\psi\rangle$ is said to have a permanent (or spontaneous) dipole moment if

$$\langle \mathbf{D} \rangle = \langle \psi | \mathbf{D} | \psi \rangle \neq 0$$

in the absence of an external electric field.
(4) We want to show that this is impossible.
Hint: Introduce $\Pi^2 = \mathbf{1}$ at various places in $\langle \psi | \mathbf{D} | \psi \rangle$ and use the results of (1), (2) and (3).

The experimental search for permanent electric dipole moments of various particles is of great experimental interest, since they can be shown to vanish under even more general conditions than the one proved above. Should such moments exist, they would give evidence for subtle effects in high energy physics.

2. In this problem you will prove *Bloch's theorem* on the form of the stationary states of a particle in a periodic potential. This result is very useful in the solid state physics of crystalline lattices.

A crystal is unchanged by a translation through a displacement

$$\mathbf{R_n} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$$

where $\mathbf{n} = (n_1, n_2, n_3)$ are three integers and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ form the edges of a unit cell of the crystal. You can think of a simple cubic crystal where these are a lattice constant a times the unit vectors along the three axes. Corresponding to such a translation there is a unitary operator

$$\widehat{U}(\mathbf{R_n}) = \exp\left(-i\widehat{\mathbf{P}} \cdot \mathbf{R_n}/\hbar\right),$$

that leaves the Hamiltonian of the system invariant:

$$\widehat{U}(\mathbf{R_n})H\widehat{U}^{-1}(\mathbf{R_n}) = H.$$

Here $\widehat{\mathbf{P}}$ is the momentum operator. Since the various $\widehat{U}(\mathbf{R_n})$'s commute with each other and with H, we can find a complete set of common eigenvectors for these operators. (Below I drop the "hats" on the operators).

Our goal is to find the general form of these common eigenvectors

$$H|\psi\rangle = E|\psi\rangle, \qquad U(\mathbf{R_n}) = \lambda(\mathbf{R_n})|\psi\rangle.$$

(1) Using $U(\mathbf{R_n})U(\mathbf{R}_\ell) = U(\mathbf{R_n} + \mathbf{R}_\ell)$ show that the eigenvalues must be of the form

$$\lambda(\mathbf{R_n}) = \exp\left(-i\mathbf{k} \cdot \mathbf{R_n}\right).$$

(2) Show that the vector $\mathbf{k}$ must be real.
(3) Show that the coordinate representation of the energy eigenfunction must satisfy:

$$\psi(\mathbf{r} - \mathbf{R_n}) = \exp\left(-i\mathbf{k} \cdot \mathbf{R_n}\right)\ \psi(\mathbf{r}).$$

Note that this is *not* a single plane-wave, the solution in the absence of any potential, but rather a function, labeled by a Bloch wavevector $\mathbf{k}$, that picks a definite phase factor when translated by a lattice vector $\mathbf{R_n}$.

3. Shankar Ex. 11.4.1 (p. 300)

4. Shankar Ex. 11.4.3 (p. 300)

5. Shankar Ex. 11.4.4 (p. 300 - 301)

6. (a) Starting with the operator definition $\mathbf{L} = \mathbf{R} \times \mathbf{P}$, show that

$$[L_x, L_y] = i\hbar L_z$$

and cyclic permutations.
(b) For $L^2 = L_x^2 + L_y^2 + L_z^2$, show that

$$\left[L^2, L_\alpha\right] = 0$$

for $\alpha = x, y, z$.

7. Shankar Ex. 12.2.1 (p. 308)

8. Shankar Ex. 12.2.3 (p. 310). You only need to do it in any one of the two ways suggested in the book.