PHYSICS 827

Home Work Assignment # 8 (final version) 10/22/2010

Due: Fri., Dec. 3, 2010 (in class).

1. In this problem we look at **Rabi Oscillations** in a simple context. These are very important in atomic, molecular and optical physics and in the field of quantum information.

(a) Argue that the most general two-level system Hamiltonian must be of the form:

$$H = \begin{pmatrix} E_0 + \epsilon & \Delta \\ \Delta^* & E_0 - \epsilon \end{pmatrix}$$

where E_0 is the average energy in the two states, ϵ determines the "asymmetry" and Δ is the (complex) "tunneling amplitude" or "coupling".

(b) Show that the energy eigenvalues are

$$E_\pm = E_0 \pm \sqrt{\epsilon^2 + |\Delta|^2}$$

and plot E_+ and E_- as functions of $|\Delta|$ on the same graph and mark various energy scales.

(c) Show that the corresponding eigenvectors are

$$|\psi_+\rangle = \cos(\theta/2)e^{-i\phi/2}|\psi_1\rangle + \sin(\theta/2)e^{+i\phi/2}|\psi_2\rangle$$

$$|\psi_-\rangle = -\sin(\theta/2)e^{-i\phi/2}|\psi_1\rangle + \cos(\theta/2)e^{+i\phi/2}|\psi_2\rangle$$

where

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is the basis in which H is written, and the angles $\theta \in [0, \pi)$ and ϕ are defined by

$$\tan\theta = |\Delta|/\epsilon \quad \text{and} \quad \Delta = |\Delta|e^{+i\phi}$$

[You could use the Pauli matrices introduced in HW # 4 to solve this problem, but that is not essential.]

(d) Let the initial state of the system be $|\psi(0)\rangle = |\psi_1\rangle$. Find the probability $P_{12}(t)$ for the system to be found in state $|\psi_2\rangle$ at a time t.

Make a careful sketch of $P_{12}(t)$, marking relevant time scales and maximum and minimum values of P_{12}.

2. In this problem we will look at a highly oversimplified caricature of the **Solar neutrino puzzle**. Very reliable models predict that essentially all the neutrinos produced in the interior of the sun are electron neutrinos ν_e via the process

$$p + p \to {}^2H + e^+ + \nu_e.$$

Further, the model makes a definite prediction for the flux of solar ν_e's incident upon the earth. Several decades of careful experiments concluded that the measured flux is only about one-half that expected from the solar model. The widely accepted explanation of this deficit is that solar models are correct, but the electron neutrinos transform into other types of neutrinos as they move from the interior of the sun to the earth. To illustrate this qualitatively, let us look at a simple theory that assumes that:
(i) there are only two types of neutrinos, the electron neutrino ν_e and the muon neutrino ν_μ. (This ignores the tau neutrino ν_τ)
(ii) and that the entire phenomenon takes place in the vacuum between the sun and earth. (This ignores propagation inside the sun).
It had long been thought that neutrinos are massless. But, if they are not, then we can work in the rest frame of the neutrinos and write down the Hamiltonian in the $\{\nu_e, \nu_\mu\}$ basis:

$$|\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\nu_\mu\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad H = \begin{pmatrix} m_e c^2 & mc^2 \\ mc^2 & m_\mu c^2 \end{pmatrix}.$$

Here m_e and m_μ are neutrino masses and the off-diagonal element m makes possible the transitions from one type of neutrino to another.

(1) Show that the mass eigenstates are

$$|\nu_1\rangle = \cos(\theta/2)|\nu_e\rangle + \sin(\theta/2)|\nu_\mu\rangle$$

$$|\nu_2\rangle = -\sin(\theta/2)|\nu_e\rangle + \cos(\theta/2)|\nu_\mu\rangle$$

with $\tan\theta = 2m/(m_e - m_\mu)$ and that the masses are

$$m_{1,2} = \frac{m_e + m_\mu}{2} \pm \sqrt{m^2 + \left(\frac{m_e - m_\mu}{2}\right)^2}$$

(2) If the state of the neutrino at time $t = 0$ in the sun is $|\psi(t)\rangle = |\nu_e\rangle$, find is the probability of observing an electron neutrino a time t later on the earth. This is the phenomenon of *neutrino oscillation*.

3. This problem uses the Pauli matrices σ_x, σ_y and σ_z and other ideas introduced in Problem 7 of HW 4. *Please review the solution of that problem in detail before working on this one, and feel free to use those results.* Consider a spin-1/2 particle with a magnetic moment

$$\mathbf{M} = \gamma\,\mathbf{S}, \quad \text{where} \quad \mathbf{S} = \frac{\hbar}{2}\,(\sigma_x, \sigma_y, \sigma_z)$$

where γ is the gyromagnetic ratio. The Hamiltonian for this system in an external magnetic field $\mathbf{B}$ is given by

$$\mathcal{H} = -\mathbf{M}\cdot\mathbf{B}.$$

Consider a uniform field that points in the (θ, ϕ)-direction, i.e.,

$$\mathbf{B} = B\,\hat{\mathbf{n}} \equiv B\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,.$$

(a) Find the state of the system $|\psi(t)\rangle$, given that it starts out in the state $|\psi(0)\rangle = |+\rangle$ in which the eigenvalue of S_z is $+\hbar/2$.

(b) At time t we measure the observable S_z. What possible values can we find and with what probabilities?

4. In this problem you will learn about **coherent states** of the quantum harmonic oscillator and explore some of their properties. These states are of great importance in quantum optics, lasers and in many areas of condensed matter physics. A *coherent state* is defined by

$$|z\rangle = C_z \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle,$$

where z is any complex number, and $|n\rangle$ is the eigenstate of the number operator $N = a^\dagger a$ with eigenvalue n, and C_z is a normalization constant that you will determine below.

(a) Show that $|z\rangle$ is a (right) eigenstate, i.e., an eigenket, of the destruction operator a and find the corresponding eigenvalue.

(b) Show that the normalization is $C_z = \exp\left(-|z|^2/2\right)$

(c) Can you find a (right) eigenstate, or eigenket, of the creation operator $a^\dagger$? If so, find it. If not, explain why not.

(d) Find $\langle z|w\rangle$ for complex z and w. Are the coherent states orthogonal?

(e) Show that the set of all coherent states $\{|z\rangle\}$ is complete basis set by demonstrating the resolution of the identity:

$$\frac{1}{\pi}\int d^2z|z\rangle\langle z| = \widehat{\mathbf{1}}.$$

where $d^2z = dx\,dy$ with $z = x + iy$.

[If you are looking for a challenge (that is optional and will not be graded): Show that the set of all coherent states is <u>over</u>complete, i.e., not all $|z\rangle$ are linearly independent.]

(f) Find the mean number $\langle z|N|z\rangle$ and the fluctuation $\langle z|N^2|z\rangle$.

(g) Calculate $\langle z_0|X|z_0\rangle$, $\langle z_0|P|z_0\rangle$ where $z_0 = x_0+iy_0$. Find the uncertainties δX and δP in the coherent state $|z_0\rangle$. What is the product $(\delta X)(\delta P)$?

(h) Find the time evolution of a coherent state: $|\psi_z(t)\rangle = \exp\left(-iHt/\hbar\right)|\psi_z(0)\rangle$, where H is the harmonic oscillator Hamiltonian, and the initial state $|\psi_z(0)\rangle = |z_0\rangle$ is a coherent state with $z_0 = \rho_0\exp(i\theta_0)$.
Show that $|\psi_z(t)\rangle$ is also a coherent state but with a time-dependent "z".

(i) Find the time evolution of the expectation values of $\langle X\rangle(t) = \langle\psi_z(t)|X|\psi_z(t)\rangle$, $\langle P\rangle(t)$ and $\langle H\rangle(t)$. Compare your results with that for a *classical* harmonic oscillator.

5. This problem builds on an example that we discussed in class. Consider the motion on a quantum particle of mass m in a one-dimensional square well: $V(x) = 0$ for $0 < x < a$ and infinite elsewhere. Denote by E_n and $|\phi_n\rangle$ the energy eigenvalues and eigenfunctions ($n = 1, 2, \ldots$).
Let the initial state of the particle be

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + |\phi_2\rangle\right)$$

We discussed $\psi(x,t) \equiv \langle x|\psi(t)\rangle$ in class.

(a) Show that the mean position of the wave packet is

$$\langle X\rangle(t) = \frac{a}{2} - \frac{16a}{9\pi^2}\cos(\omega_{21}t)$$

where $\omega_{21} = (E_2 - E_1)\hbar$.
(Hint: You may find it useful to introduce an operator $X' = X - (a/2)\widehat{1}$ for

which many of the matrix elements $\langle\phi_\ell|X'|\phi_n\rangle$ can be simply determined by symmetry.)

(b) Sketch your result for $\langle X\rangle$ as a function of time, clearly labeling relevant scales on the axes. Compare the quantum result with that for a *classical* particle moving to and fro in the well with the same frequency. Plot $x(t)$ for the latter on the same graph and comment.

(c) Find the mean energy $\langle H\rangle$, $\langle H^2\rangle$ and the the root mean squared fluctuation δH. Are these quantities time dependent? Explain.
Find the product $(\delta H)(\delta t)$ where $\delta t \simeq 1/\omega_{21}$. This is an example of a "time-energy uncertainty relation" which we will discuss later. Appearances notwithstanding, this uncertainty relation is quite different from the ones derived in class because, although there is an operator H corresponding to energy in Q.M., there is no "time" operator!