Extra Problems # 1 on Linear Algebra

Here are some interesting problems that go beyond the scope of PH 827. You should do these only if you want to be challenged beyond the regular homework.

* You will not be graded on these problems
* You do not have to master this material for PH 827. We will not use these ideas later in this class.

Defn: An operator A acting on a (finite-dim) Hilbert space $\mathbb{V}$ is called a normal operator if

$$[A, A^{+}] = 0$$

① Prove the spectral theorem for a normal operators, i.e., show that for a normal op. A we can write

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

where the eigenvectors $\{|i\rangle\}$ form an orthonormal basis in $\mathbb{V}$.

Even if you do not succeed in proving ①, you can assume it is true, and go on to the next problems.

② A normal operator is Hermetian if and only if it has real eigenvalues. Prove this.

③ A Positive operator is defined to be one that satisfies

$$\langle v | A | v \rangle \geq 0 \quad \text{for all } |v\rangle \in \mathbb{V}$$

A positive operator is necessarily Hermitian. Prove this.

④ Polar decomposition of operators: (analog of $z = |z| e^{i\theta}$ for complex numbers)

Prove that for any operator A acting on $\mathbb{V}$, there exists a unitary operator U and positive operators J and K s.t.

$$A = UJ = KU$$

where J and K are uniquely defined by $J = \sqrt{A^\dagger A}$ and $K = \sqrt{AA^\dagger}$.
Further, if A^{-1} exists, then U is unique too.

⑤ Singular Value Decomposition

Let A be any square matrix. Then there exist unitary matrices U and V and a diagonal matrix D with non-negative entries s.t.

$$A = UDV$$

The diagonal entries are called "singular values"

This generalizes the idea of "diagonalization" of Hermitian matrices via unitary transformations to arbitrary matrices and finds applications in many different fields ranging from numerical analysis to quantum computation.