Physics 730 SP08 Exam 2 Solutions

1a) $f(x) = x \quad , \quad 0 \le x \le 1$

There are several ways to do this,

20pts

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i(2\pi n)x} \qquad \text{(note } 2\pi \text{ for interval)}$$

$$c_n = \int_0^1 dx \; x\, e^{-i(2\pi n)x}$$

use $$\int_0^1 dx\; x\, e^{-ikx} = +i \frac{\partial}{\partial k} \int_0^1 dx\; e^{-ikx} = +i \frac{\partial}{\partial k} \frac{e^{-ik}-1}{-ik}$$

$$= - \left[\frac{-ie^{-ik}}{k} - \frac{e^{-ik}-1}{k^2} \right]$$

$$\xrightarrow[\substack{k=2\pi n, \\ n\neq 0}]{} \frac{i}{2\pi n}$$

$$c_0 = \int_0^1 dx \; x = \frac{1}{2}$$

$$f(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{i}{2\pi n}\right) e^{i(2\pi n)x}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{i}{2\pi n}\right) \left[e^{i(2\pi n)x} - e^{-i(2\pi n)x} \right]$$

$$= \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin[2\pi n x]}{\pi n}$$

1b) $\ddot{x} + \alpha\dot{x} + k^2 x = f(t)$

20pts

$$f(t) = \begin{cases} at & 0 \le t \le T/2 \\ a(T-t) & T/2 \le t \le T \end{cases}$$

- I caused confusion by failing to mention that the force is periodic with period T. If this were not true we could not use a Fourier series. Why did nobody ask about this?

$$x = \sum_{n=-\infty}^{\infty} c_n \exp\left[i(2\pi n)\frac{t}{T}\right]$$

$$\ddot{x} + \alpha\dot{x} + k^2 x = \sum_{n=-\infty}^{\infty} c_n \left\{ -\left(\frac{2\pi n}{T}\right)^2 + \alpha\left(\frac{i\,2\pi n}{T}\right) + k^2 \right\} \times \exp\left[i(2\pi n)\frac{t}{T}\right]$$

$$f(t) = \sum_{n=-\infty}^{\infty} d_n \exp\left[i(2\pi n)\frac{t}{T}\right]$$

$$d_n = \frac{1}{T}\int_0^T dt\; f(t) \exp\left[-i(2\pi n)\frac{t}{T}\right]$$

$n \neq 0$: $d_n = \frac{(-1+(-1)^n)}{2n^2\pi^2} \, aT$ — zero for even n

$$d_0 = \frac{aT}{4}$$

$$c_0 = \frac{aT}{4k^2}$$

$n \neq 0$: $c_n = \frac{(-1+(-1)^n)}{2n^2\pi^2} \, \frac{aT}{\left[-\left(\frac{2\pi n}{T}\right)^2 + \alpha\left(\frac{i2\pi n}{T}\right) + k^2\right]}$

2a) $$I = \int_{-\infty}^{\infty} dx\, e^{-x^2}\, \delta(x^2+x-6)$$

10 pts

zeros: $$x = \frac{1}{2}\left[-1 \pm \sqrt{1+24}\right] = \frac{1}{2}\left[-1 \pm 5\right]$$

$$x^2 + x - 6 = (x-2)(x+3)$$

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2}\left[\frac{1}{|x+3|}\delta(x-2) + \frac{1}{|x-2|}\delta(x+3)\right]$$

$$= \frac{e^{-4}}{5} + \frac{e^{-9}}{5}$$

2b) line of charge with line charge density λ along z-axis

10 pts

(i) in cylindrical coordinates we need $\rho = 0$ and we integrate $Q = \int_{-\infty}^{\infty} dz \int_0^{\pi} d\theta \int_0^{\infty} d\rho\, \rho\, q(\rho,\theta,z)$ ← charge density

$$q(\rho,\theta,z) = \frac{\lambda}{\pi}\frac{\delta(\rho)}{\rho} \quad \text{works}$$

– remember $\int_0^{\infty} d\rho\, \delta(\rho) = \frac{1}{2}$

This leads to $Q = \lambda \int_{-\infty}^{\infty} dz$ which is correct

(ii) spherical co-ordinates

- here we need $\Theta = 0$ and $\Theta = \pi$

  + uniform density along z-axis

- I will change variables, starting with $\rho = \lambda\,\delta(x)\,\delta(y)$

  $$\rho = \lambda\,\delta(r\sin\Theta\cos\phi)\,\delta(r\sin\Theta\sin\phi).$$

  This seems to lead nowhere, because ρ does not have any ϕ dependence and $[\delta(r\sin\Theta)]^2$ is too singular,

  But we can use $\delta(\sin\Theta)$ to make ρ.

Try $$\rho(r,\Theta,\phi) = \frac{\lambda}{2\pi}\,\frac{\delta(\sin\Theta)}{r^2\sin\Theta}$$

$$\Rightarrow\quad Q = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} r^2 dr\ \frac{\lambda}{2\pi}\,\frac{\delta(\sin\theta)}{r^2\sin\theta}$$

$$= \lambda\int_0^{\pi} d\Theta \int_0^{\infty} dr\ \delta(\sin\Theta)$$

- vanishes except at $\Theta = 0$ and π and is constant for all r at these two angles with charge per unit length λ

3a) $\sum_{p=0}^{\infty} (-1)^p \frac{2p+1}{x^2+(2p+1)^2} = f(x)$

20 pts

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \sum_{p=0}^{\infty} (-1)^p (2p+1) \underbrace{\int_{-\infty}^{\infty} dx \frac{e^{-ikx}}{x^2+(2p+1)^2}}_{I}$$

$k > 0$, must close downwards and find residue from pole at $x = -i(2p+1)$

$$I = \underset{\text{clockwise}}{-} 2\pi i \frac{e^{-(2p+1)k}}{-2i(2p+1)} = \frac{\pi}{2p+1} e^{-(2p+1)k}$$

$k < 0$, close upwards with pole at $x = i(2p+1)$

$$I = 2\pi i \frac{e^{(2p+1)k}}{2i(2p+1)} = \frac{\pi}{2p+1} e^{(2p+1)k}$$

So $I = \frac{\pi}{2p+1} e^{-(2p+1)|k|}$

$$\tilde{f}(k) = \frac{\pi}{\sqrt{2\pi}} \sum_{p=0}^{\infty} (-1)^p e^{-|k|} \left[e^{-2|k|}\right]^p$$

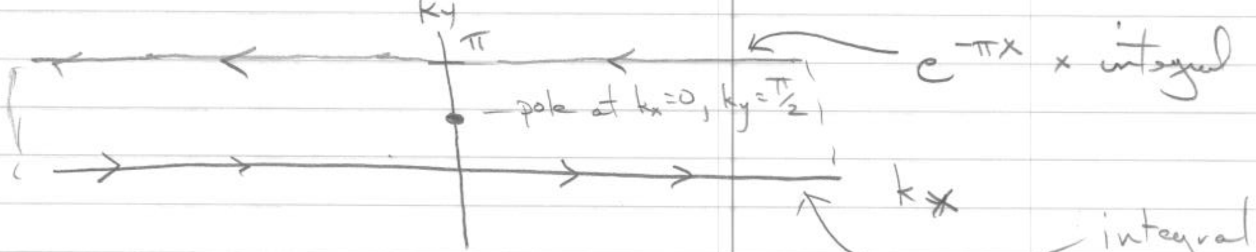
$$= \frac{1}{\sqrt{2\pi}} \frac{\pi e^{-|k|}}{1+e^{-2|k|}} = \frac{\pi/\sqrt{2\pi}}{e^{|k|}+e^{-|k|}} = \frac{\pi/\sqrt{2\pi}}{e^{k}+e^{-k}}$$

$$F(x) = \frac{1}{2}\int_{-\infty}^{\infty} dk \, \frac{e^{ikx}}{e^{k} + e^{-k}}$$

We need to use the contour the book uses for integrating $\frac{1}{\cosh(x)}$, but here we let

$k \to k_x + ik_y$ and note for $k_y = \pi$

$$\frac{e^{i(k_x+ik_y)x}}{e^{(k_x+ik_y)} + e^{-(k_x+ik_y)}} \xrightarrow[k_y=\pi]{} \frac{e^{-\pi x} \, e^{ik_x x}}{-\left[e^{k_x} + e^{-k_x}\right]}$$



$$\text{residue} = \lim_{z \to \frac{i\pi}{2}} \frac{e^{ixz}}{2\frac{d}{dz}(e^{z}+e^{-z})} = \frac{e^{-\pi x/2}}{2(e^{i\pi/2} - e^{-i\pi/2})} = \frac{e^{-\pi x/2}}{4i}$$

$$(1 + e^{-\pi x}) \, F(x) = 2\pi i \, \frac{e^{-\pi x/2}}{4i} = \frac{\pi}{2} e^{-\pi x/2}$$

$$\boxed{F(x) = \frac{\pi \, e^{-\pi x/2}}{2(1 + e^{-\pi x})} = \frac{\pi}{4\cosh(\pi x/2)}}$$

$$x(t) = i\Omega\,\theta(t)\,e^{-\alpha t}\left\{ \frac{e^{i\Omega t}}{2\Omega(\omega_0^2-\alpha^2-\Omega^2)} - \frac{e^{-i\Omega t}}{2\Omega(\omega_0^2-\alpha^2-\Omega^2)} + \frac{e^{i\sqrt{''}\,t}}{2\sqrt{''}\,(\Omega^2-\omega_0^2+\alpha^2)} - \frac{e^{-i\sqrt{''}\,t}}{2\sqrt{''}\,(\Omega^2-\omega_0^2+\alpha^2)} \right\}$$

$$x(t) = \Omega\,\theta(t)\,e^{-\alpha t}\left\{ \frac{\sin(\Omega t)}{\Omega(\Omega^2+\alpha^2-\omega_0^2)} - \frac{\sin\left(\sqrt{\omega_0^2-\alpha^2}\,t\right)}{\sqrt{\omega_0^2-\alpha^2}\,(\Omega^2+\alpha^2-\omega_0^2)} \right\}$$

$$x(t) = \theta(t)\,\frac{\Omega e^{-\alpha t}}{\Omega^2+\alpha^2-\omega_0^2}\left\{ \frac{\sin(\Omega t)}{\Omega} - \frac{\sin\left(\sqrt{\omega_0^2-\alpha^2}\,t\right)}{\sqrt{\omega_0^2-\alpha^2}} \right\}$$

3b) $\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = F(t)$, $\omega_0 > \alpha$

20 pts

$$F(t) = e^{-\alpha t}\sin(\Omega t)\,\Theta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \tilde{F}(\omega)\, e^{-i\omega t}$$

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \tilde{x}(\omega)\, e^{-i\omega t}$$

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \left[-\omega^2 - 2i\alpha\omega + \omega_0^2\right]\tilde{x}(\omega)\, e^{-i\omega t}$$

$$\sqrt{2\pi}\,\tilde{F}(\omega) = \int_{-\infty}^{\infty} dt\, F(t)\, e^{i\omega t} = \frac{1}{2i}\int_{0}^{\infty} dt\, e^{-\alpha t}\left[e^{i\Omega t} - e^{-i\Omega t}\right]e^{i\omega t}$$

$$= \frac{1}{2i}\left[\frac{e^{i\Omega t - \alpha t + i\omega t}}{i(\Omega+\omega)-\alpha} - \frac{e^{-i\Omega t - \alpha t + i\omega t}}{-i(\Omega-\omega)-\alpha}\right]_0^{\infty}$$

$$= \frac{1}{2i}\left[\frac{1}{\alpha - i(\Omega+\omega)} - \frac{1}{\alpha + i(\Omega-\omega)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega+\Omega+i\alpha} - \frac{1}{\omega-\Omega+i\alpha}\right]$$

$$\tilde{x}(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\frac{1}{\omega+\Omega+i\alpha} - \frac{1}{\omega-\Omega+i\alpha}\right] \times \left[-\omega^2 - 2i\alpha\omega + \omega_0^2\right]^{-1}$$

Before transforming back to $x(t)$, rewrite last denominator.

poles of $\omega^2 + 2i\alpha\omega - \omega_0^2$:

$$\omega = \frac{1}{2}\left[-2i\alpha \pm \sqrt{-4\alpha^2 + 4\omega_0^2}\right] = -i\alpha \pm \sqrt{\omega_0^2 - \alpha^2}$$

$$-\omega^2 - 2i\alpha\omega + \omega_0^2 = -\left(\omega + i\alpha + \sqrt{\omega_0^2-\alpha^2}\right)\left(\omega + i\alpha - \sqrt{\omega_0^2-\alpha^2}\right)$$

$$x(t) = \frac{\Omega}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{(\omega+\Omega+i\alpha)(\omega-\Omega+i\alpha)(\omega+\sqrt{\omega_0^2-\alpha^2}+i\alpha)(\omega-\sqrt{\omega_0^2-\alpha^2}+i\alpha)}$$

Note: all poles below real axis, so if $t<0$, $x(t) = 0$ because contour must be closed upwards.

$$t>0: \quad x(t) = i\Omega \left\{ \frac{e^{-i(-\Omega-i\alpha)t}}{(-2\Omega)(\sqrt{\omega_0^2-\alpha^2}-\Omega)(-\sqrt{\omega_0^2-\alpha^2}-\Omega)} \right.$$

$$+ \frac{e^{-i(\Omega-i\alpha)t}}{(2\Omega)(\sqrt{\omega_0^2-\alpha^2}+\Omega)(\Omega-\sqrt{\omega_0^2-\alpha^2})}$$

$$+ \frac{e^{-i(-\sqrt{''}-i\alpha)t}}{(\Omega+\sqrt{''})(\sqrt{''}-\Omega)(-2\sqrt{''})}$$

$$\left. + \frac{e^{-i(\sqrt{''}-i\alpha)t}}{(\Omega+\sqrt{''})(\sqrt{''}-\Omega)(2\sqrt{''})} \right\}$$