States

Classically, for one particle

$$x(t)$$

completely specifies a 'state'.

In quantum mechanics the state of a one-particle 'system' is:

$$|\psi(t)\rangle$$

Wave function:

$$\psi(x,t) = \langle x | \psi(t) \rangle$$

Hilbert space

Space of vectors with a scalar product.

Vectors $\vec{v}$

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

$\vec{v} + \vec{w}$ is also a vector.

normalized vector $|\vec{v}|^2 = \vec{v} \cdot \vec{v} = 1$

basis: $\hat{x}, \hat{y}, \hat{z}$; $\hat{x} \cdot \hat{x} = 1$

$$\vec{v} = v_1 \hat{x} + v_2 \hat{y} + v_3 \hat{z}$$

States $|\psi\rangle$

$$\langle \phi | \psi \rangle = \int_{-\infty}^{\infty} dx \, \phi^*(x) \psi(x)$$

$|\phi\rangle + |\psi\rangle$ is also a state

normalized basis $\langle \phi_n | \phi_m \rangle = \delta_{mn}$

Operators

states $\rightarrow$ states

$$\hat{O}|\phi\rangle = |\psi\rangle$$

- in general, an operator $\hat{O}$ 'acts on' a state to produce a completely different state

Eigenstates $|\phi_n\rangle$

$$\hat{O}|\phi_n\rangle = o_n|\phi_n\rangle$$

$\uparrow$ eigenvalue

If $\hat{O} = \hat{O}^+$ (Hermitian), its eigenstates form a basis:

$$|\psi\rangle = \sum_n b_n |\phi_n\rangle$$

$$b_n = \langle\phi_n|\psi\rangle$$

$\left\{ \begin{array}{l} \text{analog of:} \\ v_x = \hat{i}\cdot\vec{v} \\ \quad = \hat{x}\cdot\vec{v} \end{array} \right.$

Scalar products with operators and 'adjoint' operators

$$\langle \phi | \hat{O} | \psi \rangle = \langle \phi | O\psi \rangle$$

$$= \langle O^{+}\phi | \psi \rangle$$

O^{+} - adjoint, defined so that scalar products are equal

Momentum example

$$\langle \phi | \hat{P} | \psi \rangle = \int_{-\infty}^{\infty} dx \; \phi^{*}(x) \left[-i\hbar \frac{d\psi}{dx} \right]$$

$$\langle \phi | P\psi \rangle$$

$$= \int_{-\infty}^{\infty} dx \left[i\hbar \frac{d\phi^{*}}{dx} \right] \psi(x)$$

$$\langle P^{+}\phi | \psi \rangle \rightarrow = \int_{-\infty}^{\infty} dx \left[-i\hbar \frac{d\phi}{dx} \right]^{*} \psi(x)$$

$\hookrightarrow$ $P^{+} = P$ - momentum is Hermitian

Eigenstates of Hermitian Ops.

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

- energy eigenstates

$$\hat{P}|p\rangle = p|p\rangle$$

- momentum eigenstates

$$\hat{X}|x\rangle = x|x\rangle$$

- position eigenstates

Completeness

$$\hat{1} = \sum_n |\phi_n\rangle\langle\phi_n|$$

$$= \frac{1}{\hbar}\int dp\ |p\rangle\langle p|$$

$$= \int dx\ |x\rangle\langle x|$$

Using a basis

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

$$\hat{1} = \sum_n |\phi_n\rangle\langle\phi_n|$$

$$|\psi\rangle = \hat{1}|\psi\rangle$$

$$= \sum_n |\phi_n\rangle\langle\phi_n|\psi\rangle$$

$$\langle\phi_n|\psi\rangle = \langle\phi_n|\hat{1}|\psi\rangle$$

$$= \langle\phi_n|\left[\int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\right]|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx\, \langle\phi_n|x\rangle\langle x|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx\, \phi_n^*(x)\,\psi(x)$$

If $|\psi\rangle = \sum_n b_n |\phi_n\rangle$,

and $\langle\phi_m|\phi_n\rangle = \delta_{mn}$ (basis)

$$b_n = \langle\phi_n|\psi\rangle$$

If $\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$,

$$\hat{H}|\psi\rangle = \hat{H}\sum_n b_n|\phi_n\rangle$$

$$= \sum_n b_n E_n |\phi_n\rangle$$

Use eigenstates to compute action of an operator.

$$e^{-i\hat{H}t/\hbar}|\psi\rangle$$

$$= \sum_n b_n e^{-iE_nt/\hbar}|\phi_n\rangle$$

Formal solution of Schrödinger Eq.

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

IF $\frac{d}{dt}\hat{H} = 0$:

$$|\psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar} |\psi(t_0)\rangle$$

Given the state at t_0, $|\psi(t_0)\rangle$,

$$|\psi(t_0)\rangle = \sum_n |\phi_n\rangle \underbrace{\langle\phi_n|\psi(t_0)\rangle}_{\#\text{s},\ b_n}$$

$$e^{-i\hat{H}(t-t_0)/\hbar} |\psi(t_0)\rangle$$

$$= \sum_n b_n e^{-iE_n(t-t_0)/\hbar} |\phi_n\rangle.$$

Use $\hat{1}$

$$\hat{1} = \sum_n |\phi_n\rangle\langle\phi_n|$$

$$|\psi\rangle = \sum_n |\phi_n\rangle\langle\phi_n|\psi\rangle$$

$$\hat{1} = \int dx \; |x\rangle\langle x|$$

$$|\psi\rangle = \int dx \; |x\rangle\langle x|\psi\rangle$$

$$= \int dx \; |x\rangle \, \psi(x)$$

$$\hat{1} = \frac{1}{\hbar}\int dp \; |p\rangle\langle p|$$

$$|\psi\rangle = \frac{1}{\hbar}\int dp \; |p\rangle\langle p|\psi\rangle$$

$$= \frac{1}{\hbar}\int dp \; |p\rangle \, \Phi(p)$$

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi}} e^{ipx/\hbar} = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

$$\langle x | \psi \rangle = \psi(x)$$

$$\langle p | \psi \rangle = \langle p | \left\{ \int dx\, |x\rangle\langle x| \right\} |\psi\rangle$$

$$= \int dx\, \langle p | x \rangle \langle x | \psi \rangle$$

$$= \frac{1}{\sqrt{2\pi}} \int dx\, e^{-ipx/\hbar}\, \psi(x)$$

$$\equiv \tilde{\psi}(p)$$

This is a Fourier transform.