Radial Schrödinger Eq.

Use $\vec{L} = \vec{R} \times \vec{P}$ to compute L^2.

$$\hat{P}^2 = \hat{P}_r^{\,2} + \frac{L^2}{R^2} \quad ; \quad [L^2, R^2] = 0$$

$$\hat{P}_r = \frac{1}{2}\left[\frac{1}{R}\left(\vec{R}\cdot\vec{P}\right) + \left(\vec{P}\cdot\vec{R}\right)\frac{1}{R}\right] \quad [L^2, P_r^2] = 0$$

in position rep.: $\hat{P}_r \to -i\hbar \frac{1}{r}\frac{\partial}{\partial r} r$

$$\hat{H}_{Free} = \frac{\hat{P}_r^{\,2}}{2m} + \frac{\hat{L}^2}{2mr^2} \quad \text{in spherical coords.}$$

$$\hat{H}_{free} = -\frac{1}{2m}\nabla^2$$

$$= -\frac{1}{2m}\left\{\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right\}$$

The time-independent Schrödinger eq. for a Free particle is:

$$\frac{1}{2m}\left(\hat{P}_r^2 + \frac{\hat{L}^2}{r^2}\right)\phi_{klm} = E_{klm}\,\phi_{klm}$$

$$\phi_{klm} = R_{kl}(r)\,Y_l^m(\theta,\phi)$$

spherical harmonic

$$\hat{L}^2 Y_l^m = l(l+1)\hbar^2 Y_l^m$$

$$\left[-\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r\right) + \frac{l(l+1)}{r^2}\right]R_{kl}(r) = 2mE\,R_{kl}(r)$$

Let $E = \frac{k^2}{2m}$, $x \equiv kr$

$$\frac{d^2}{dx^2}R(x) + \frac{2}{x}\frac{dR}{dx} + \left[1 - \frac{l(l+1)}{x^2}\right]R(x) = 0$$

$\hookrightarrow$ Bessel's eq.

$$R_{kl}(r) = j_l(kr)$$

$\hookrightarrow$ spherical Bessel's Fnc.

$$j_0(x) = \frac{\sin(x)}{x} \qquad n_0(x) = -\frac{\cos(x)}{x}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad n_1(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x}$$

$$\phi_{k\ell m}(r,\theta,\phi) = j_\ell(kr)\, Y_\ell^m(\theta,\phi)$$

$$\langle k\ell m | k'\ell' m' \rangle = \delta_{\ell\ell'}\,\delta_{mm'}\,\frac{\pi}{2k^2}\,\delta(k-k')$$

Particle confined in a sphere

$$V(r) = \begin{cases} 0 & r<R \\ \infty & r>R \end{cases}$$

$$\to \psi(R,\theta,\phi,t) = 0$$

$$\to j_\ell(kR) = 0 \Rightarrow$$ determines allowed values of k

$\boxed{\ell = 0}$

$$j_0(kR) = \frac{\sin(kR)}{kR} = 0$$

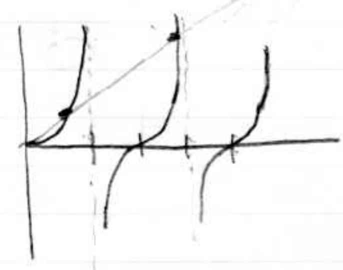
$$\sin(kR) = 0$$

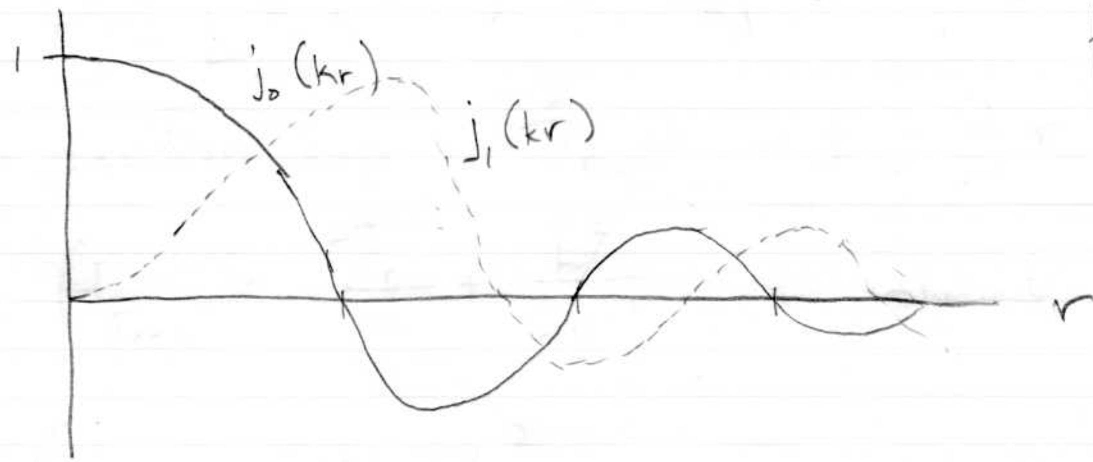
$$k = \frac{n\pi}{R}, \quad n = 1, 2, 3, \ldots$$

$\boxed{\ell = 1}$

$$j_1(kR) = \frac{\sin(kR)}{k^2R^2} - \frac{\cos(kR)}{kR} = 0$$

$$\left.\frac{\sin(kR)}{kR} = \cos(kR)\ \right\} \quad x = \tan(x)$$





$$j_\ell(x) \xrightarrow[x \to 0]{} x^\ell$$