Basic Linear Algebra for QM

linear vector space : $|n\rangle$, $|k\rangle$

discrete continuous

$\alpha|n_1\rangle + \beta|n_2\rangle$ is in space, etc.

scalar product exists $\langle m|n\rangle = \delta_{mn}$

discrete

$$|\phi\rangle = \sum_n a_n |n\rangle , \quad |\psi\rangle = \sum_n b_n |n\rangle$$

$$\langle\psi|\phi\rangle = \sum_{m,n} \langle m| b_m^* a_n |n\rangle$$

$$= \sum_n b_n^* a_n$$

continuous

continuous indices $\langle k|k'\rangle = \delta(k-k')$

$$|\phi\rangle = \int dk\, a(k)|k\rangle , \quad |\psi\rangle = \int dk\, b(k)|k\rangle$$

$$\langle\psi|\phi\rangle = \int dk_1 \int dk_2 \langle k_1| b^*(k_1) a(k_2) |k_2\rangle$$

$$= \int dk\, b^*(k) a(k)$$

Schwarz inequality $|\langle\psi|\phi\rangle|^2 \le |\psi|^2 |\phi|^2$

Linear operators

$$\mathbb{1} = \sum_n |n\rangle\langle n| = \int dk\, |k\rangle\langle k|$$

$$A|n\rangle = \sum_m |m\rangle\langle m|A|n\rangle$$

$$= \sum_m A_{mn} |m\rangle$$

adjoint

$$\langle n|A^\dagger = \sum_m \langle n|A^\dagger|m\rangle\langle m|$$

hermitean conjugate

$$= \sum_m A^\dagger_{nm} \langle m|$$

$$= \sum_m A^*_{mn} \langle m| \quad \text{from above}$$

$$\Rightarrow A^\dagger_{nm} = A^*_{mn} \;; \quad A^\dagger = (A^T)^*$$

A^T is transpose

Note: $|\psi\rangle = \mathbb{1}|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle$

$$= \sum_n b_n |n\rangle$$ - representation in $|n\rangle$ basis

$$b_n = \langle n|\psi\rangle$$

hermitian ops $\quad A = A^{\dagger}$

antihermitian ops $\quad A = -A^{\dagger}$

IF A is hermitian:

$$\left.\begin{aligned} A|i\rangle &= a_i|i\rangle \\ \mathbb{1} &= \sum_i |i\rangle\langle i| \end{aligned}\right\}\ \text{a new basis}$$

IF B is also hermitian:

$$B|j\rangle = b_j|j\rangle$$

$[A,B] = 0 \implies |i\rangle$s and $|j\rangle$s are the same

$[A,B] \neq 0 \implies |i\rangle$'s and $|j\rangle$s are different

→ We can 'diagonalize' A or B, but not both.

Note: $\langle i|A|j\rangle = A_{ij} = a_i\delta_{ij}$

– matrix is diagonal in eigenstate representation