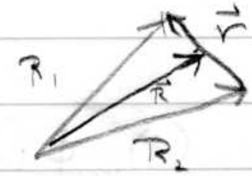
$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|) \leftarrow \text{Galilean symmetries}$$

Center-of-Mass Co-ordinates



$$\vec{R} = (m_1\vec{R}_1 + m_2\vec{R}_2) / (m_1 + m_2)$$

$$\vec{r} = \vec{R}_2 - \vec{R}_1$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \, , \quad \vec{p} = \frac{m_1\vec{p}_2 - m_2\vec{p}_1}{m_1 + m_2}$$

$$H = \frac{P^2}{2M} + \left[\frac{p^2}{2\mu} + V(r) \right] = H_{cm} + H_{rel}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \, , \quad M = m_1 + m_2$$

$\llcorner$ reduced mass $\qquad \llcorner$ total mass

$$[\hat{r}_{1i}, \hat{p}_{1j}] = [\hat{r}_{2i}, \hat{p}_{2j}] = i\hbar\,\delta_{ij}$$

$$\hookrightarrow [\hat{r}_i, \hat{p}_j] = [\hat{R}_i, \hat{P}_j] = i\hbar\,\delta_{ij}$$

$$\hat{H} = \hat{H}_{cm} + \hat{H}_{rel}$$

$$\hat{H}_{cm} = \frac{\hat{P}^2}{2M}$$

$$H_{rel} = \frac{\hat{p}^2}{2\mu} + V(r)$$

$$|\psi\rangle = |\phi_{cm}\rangle |\phi_{rel}\rangle$$

$$\langle \vec{r}_1, \vec{r}_2 | \psi\rangle = \langle \vec{r}, \vec{R} | \psi \rangle = \phi_{cm}(\vec{R})\, \phi_{rel}(\vec{r})$$

$$\hat{H}_{cm} |\phi_{cm}\rangle = E_{cm} |\phi_{cm}\rangle$$

$$\hat{H}_{rel} |\phi_{rel}\rangle = E_{rel} |\phi_{rel}\rangle$$

$$\left\{ \begin{aligned} \frac{\hat{P}^2}{2M} |\phi_{cm}\rangle &= \frac{\hbar^2 K^2}{2M} |\phi_{cm}\rangle \\ \phi_{cm}(\vec{R}) &= \left(\frac{1}{2\pi}\right)^{3/2} e^{i\vec{K}\cdot\vec{R}} \\ &= j_{l_c}(KR)\, Y_{l_c}^{m_c}(\theta_c, \varphi_c) \end{aligned} \right.$$

We usually ignore CM wavefunction and work in the CM frame where $\vec{K} = 0$.

boosts: $\vec{v} \to \vec{v} + \vec{v}_0$, $\vec{x} \to \vec{x} + \vec{v}_0 t$

$$\vec{p} \to \vec{p} + m\vec{v}_0 \;;\; \vec{P} \to \vec{P} + M\vec{v}_0,\ \vec{p} \to \vec{p}$$

$$\vec{R} \to \vec{R} + \vec{v}_0 t,\ \vec{r} \to \vec{r}$$

We are left with:

rel. motion $\Big\}$ $$\left[\frac{\hat{P}^2}{2\mu} + V(r)\right]\varphi = E\varphi$$

$$\hat{H}\varphi = \left[\frac{\hat{P}_r^2}{2\mu} + \frac{\hat{L}^2}{2\mu r^2} + V(r)\right]\varphi = E\varphi$$

$$\varphi = R(r)\,Y_\ell^m(\theta,\phi)$$

$$\left[\frac{\hat{P}_r^2}{2\mu} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + V(r)\right]R = ER$$

$$V_{eff} = V(r) + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$$

$u(r) = rR$; $u(0) = 0$ - required for solution to Sch., avoids $\delta^3(\vec{r})$

$$\hat{P}_r^2 R = -\hbar^2 \frac{1}{r}\frac{\partial^2}{\partial r^2}u$$

$$\boxed{\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + V_{eff}(r)\right]u(r) = E\,u(r)}$$

h.o. $V_{eff}(r) = \frac{1}{2}\mu\omega^2 r^2 + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$

$$R_{k\ell}(r) = \frac{1}{r} u_{k\ell}(r)$$

$$\varepsilon_{k\ell} = \frac{2\mu E_{k\ell}}{\hbar^2}$$

$$\left[\frac{d^2}{dr^2} - \beta^4 r^2 - \frac{\ell(\ell+1)}{r^2} + \varepsilon_{k\ell}\right] u_{k\ell}(r) = 0$$

$$u_{k\ell}(r) = e^{-\beta^2 r^2/2} \, y_{k\ell}(r)$$

$$E_{k\ell} = \hbar\omega\left(k + \ell + \frac{3}{2}\right)$$

$$k = 0, 2, 4, \ldots$$

produces same degeracies as $(n_x + n_y + n_z + \frac{3}{2})\hbar\omega$