Towards relativistic Quantum Mechanics

The laws of physics are the same in all <u>inertial</u> frames of reference.

$\Rightarrow$ $x(t)$, $p(t)$; $|\psi(t)\rangle$ - These can vary, but there must be a connection:

Galilean / non-relativistic $t'=t$ $\left.\right\}$ $x(t), p(t) \longrightarrow x'(t), p'(t)$

$$|\psi(t)\rangle \longrightarrow |\psi'(t)\rangle$$

$\rightarrow$ = rotation, translation, <u>boost</u>

parity?, time reversal?

<u>boost</u>: $x(t) \longrightarrow x(t) + vt = x'(t)$

$\ddot{x}' = \ddot{x}$ - everybody agrees on acceleration, so this is the quantity relevant for <u>laws</u> of motion

Nonrelativistic

Time is invariant and boosts are <u>kinematic</u> - don't need to know forces

Rotations, translations and boosts are given by a unitary transformation:

$$|\psi\rangle \longrightarrow U|\psi\rangle = |\psi'\rangle$$

$$\mathcal{O} \longrightarrow U\mathcal{O}U^{-1} = \mathcal{O}'$$

$$\langle\psi_1'|\mathcal{O}'|\psi_2'\rangle = \langle\psi_1|U^{-1}[U\mathcal{O}U^{-1}]U|\psi_2\rangle$$
$$= \langle\psi_1|\mathcal{O}|\psi_2\rangle$$

$\Rightarrow$ probabilities are the same for both observors

We are insured that 'the physics' is the same in all inertial frames by relating all elements in our theory that change from one frame to another by unitary transformations.

Relativity:

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

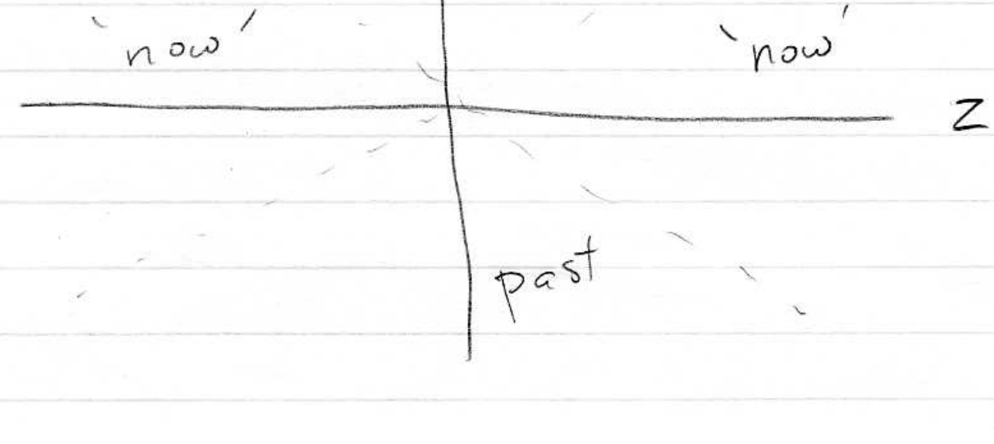
↑ invariant length

$$ds'^2 = ds^2$$ – same in all frames (scalar)

Consider a boost in the x direction:

$$x' = \frac{x + vt}{(1 - v^2/c^2)^{1/2}}, \quad y' = y, \quad z' = Z$$

$$t' = \frac{t + vx/c^2}{(1 - v^2/c^2)^{1/2}}$$ – different observors do not agree on time!



4-vectors

Nonrelativistically we describe positions with vectors, $\vec{r}$), and time is a separate quantity. Since $t' = t$ under all transformations from one inertial frame to another, there is no reason to combine $\vec{r}$ and t.

Relativistically $ct' = \frac{ct + vx/c}{\sqrt{1 - v^2/c^2}}$, so

time and space co-ordinates mix under boosts.

$$\Rightarrow \quad x^\mu = (ct, x, y, z) \quad \Big\} \text{ Four-vector}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh(\phi) & \sinh(\phi) & 0 & 0 \\ \sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\cosh(\phi) = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \sinh(\phi) = \frac{v/c}{\sqrt{1 - v^2/c^2}}$$

check: $\cosh^2(\phi) - \sinh^2(\phi) = 1$ ✓

contravariant vector $\quad$ covariant vector

In general : $(x')^{\mu} = \Lambda^{\mu\nu} x_{\nu}$

We 'contract' upper indices with lower indices (not $x^{\mu}x^{\mu}$), so that :

$$x_{\mu}x^{\mu} = c^2t^2 - \vec{x}^{\,2} \qquad \text{- scalar product}$$

need a minus sign

$$x_{\mu} = (ct, -\vec{x}) \quad , \quad x^{\mu} = (ct, \vec{x})$$

$$\begin{aligned} x_{\mu}x^{\mu} &= x_0x^0 + x_1x^1 + x_2x^2 + x_3x^3 \\ &= (ct)(ct) + (-x)(x) + (-y)(y) + (-z)(z) \\ &= c^2t^2 - \vec{x}^{\,2} \end{aligned}$$

$$x_{\mu} = g_{\mu\nu}x^{\nu} \quad , \quad x^{\mu} = g^{\mu\nu}x_{\nu}$$

$\boxed{\text{metric tensor}}$ -use to raise + lower indices

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- avoids need for ict
- another convention uses $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

The equation on p.4 is :

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} = \Lambda^{\mu\alpha}g_{\alpha\nu}x^{\nu}$$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Lambda^{\mu\alpha} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ \sinh\phi & -\cosh\phi & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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$$\cosh(\phi) = \frac{1}{\sqrt{1 - v^2/c^2}} \; ; \; \sinh(\phi) = \frac{v/c}{\sqrt{1 - v^2/c^2}}$$

check: $\cosh^2(\phi) - \sinh^2(\phi) = 1 \quad \checkmark$

Note that 'lengths' can be negative.

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

$ds^2 > 0$ - time-like

$ds^2 < 0$ - space-like

$ds^2 = 0$ - null or light-like

derivatives: $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \partial_1, \partial_2, \partial_3)$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

note the signs, 'opposite' other 4-vectors

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

d'Alembertian: $\Box = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$

Energy-momentum: $p^\mu = \left(\frac{1}{c}E, \vec{p}\right)$, $p_\mu = \left(\frac{1}{c}E, -\vec{p}\right)$

$$p^2 = \frac{1}{c^2}E^2 - \vec{p}\cdot\vec{p} = m^2c^2$$

Typically let $c = 1$: $p^2 = E^2 - \vec{p}^{\,2} = m^2$

$$p \cdot x = Et - \vec{p}\cdot\vec{r}$$

Klein-Gordon Equation

First quantization: $E \to i\hbar \frac{\partial}{\partial t}$, $\vec{p} \to -i\hbar\vec{\nabla}$

$$\frac{1}{c^2}E^2 - \vec{p}^{\,2} = m^2c^2 \longrightarrow \boxed{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\phi + \frac{m^2c^2}{\hbar^2}\phi = 0}$$

Klein-Gordon equation

$$\text{or: } (\Box + m^2)\,\phi = 0$$

Need a number current & density to provide probabilities that are conserved:

$$\text{need } \partial_\mu j^\mu = 0$$

$$j^\mu = (\rho, \vec{j}) \quad ; \quad \rho = \#\text{ density}$$

Use: $\rho = \frac{i\hbar}{2m}\left(\phi^* \frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) = \frac{i\hbar}{m}\phi^*\overleftrightarrow{\partial_0}\phi$

$$\vec{j} = \frac{i\hbar}{2m}\left(-\phi^*(\vec{\nabla}\phi) + (\vec{\nabla}\phi^*)\phi\right) = \frac{i\hbar}{m}\phi^*(-\overleftrightarrow{\nabla})\phi$$

Check: $\partial_\mu j^\mu = \frac{i\hbar}{2m}\left(\phi^*\Box\phi - (\Box\phi^*)\phi\right) = 0$

use Klein-Gordon equation

Problem: ρ can be negative

- cannot interpret as a probability density

Cannot use Klein-Gordon eq. as a single particle equation.

After 'second quantization', ϕ is a Field.

Complex ϕ represents charged particles, $\rho < 0$ corresponds to a positive probability for negative charge.

2nd Quantization:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k) e^{-ikx} + a^\dagger(k) e^{ikx} \right]$$

$a^\dagger(k)$ - creates a particle $|k\rangle = a_k^\dagger |0\rangle$

$a(k)$ - annihilates a particle

$|0\rangle$ - vacuum (bare)

Problem: $E = \pm \left(\vec{p}^2 c^2 + m^2 c^4 \right)$

- negative energy solutions