Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}K\hat{X}^2 \quad , \quad \omega_0 = \sqrt{\frac{K}{m}}$$ natural frequency

$$\hat{H}|\phi\rangle = E|\phi\rangle$$

$$\xrightarrow[\text{position}]{} \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2\right]\phi(x) = E\phi(x)$$

I) $\underline{\underline{E > 0}}$; $\langle\phi|\hat{H}|\phi\rangle = \int dp\ \frac{1}{2m}p^2|\tilde{\phi}(p)|^2 + \frac{1}{2}m\omega_0^2\int dx\ x^2|\phi(x)|^2$

> 0 > 0

II) $V(x) = V(-x) \Rightarrow \phi(-x) = \pm\phi(x)$ definite parity

III) spectrum is discrete : true for any bd. states in a potential well

Algebraic Method

Write $\hat{H}$ as a quadratic & 'Factor':

$$\boxed{\beta = \sqrt{m\omega_0}} := (m.K)^{1/4}$$

$$\tilde{X} = \beta\hat{X} \quad , \quad \tilde{P} = \frac{1}{\beta}\hat{P}$$

simplify algebra later

$$[\tilde{X}, \tilde{P}] = i$$

$$\boxed{\hat{H} = \omega_0\tilde{H} \quad \tilde{H} = \frac{1}{2}\left(\tilde{X}^2 + \tilde{P}^2\right)}$$

$$\hat{a} \equiv \frac{1}{\sqrt{2}}\left(\tilde{X} + i\tilde{P}\right) \; ; \; a^{+} = \frac{1}{\sqrt{2}}\left(\tilde{X} - i\tilde{P}\right)$$

or $\tilde{X} = \frac{1}{\sqrt{2}}\left(a^{+} + a\right)$; $\tilde{P} = \frac{i}{\sqrt{2}}\left(a^{+} - a\right)$

$$\left[a, a^{+}\right] = \frac{1}{2}\left[\tilde{X} + i\tilde{P},\, \tilde{X} - i\tilde{P}\right]$$

$$= \frac{i}{2}\left[\tilde{P}, \tilde{X}\right] - \frac{i}{2}\left[\tilde{X}, \tilde{P}\right]$$

$$\boxed{\left[a, a^{+}\right] = 1}$$

rewrite everything using the factors a and a^{+}

$$a^{+}a = \frac{1}{2}\left(\tilde{X}^{2} + \tilde{P}^{2} - 1\right)$$

$$\boxed{\hat{H} = \omega_0 \tilde{H} = \omega_0\left(a^{+}a + \frac{1}{2}\right)}$$

$N \equiv a^{+}a$ - will see this is a number operator

$$\boxed{\hat{H} = \omega_0\left(N + \frac{1}{2}\right)}$$

Use a and $a^\dagger$ to construct eigenstates:

to prove $\Big\}$ $$\hat{H}|\phi_n\rangle = \omega_0\left(n+\tfrac{1}{2}\right)|\phi_n\rangle$$

$$N|\phi_n\rangle = n|\phi_n\rangle \qquad n = 0, 1, 2, \ldots$$

$$|\phi_n\rangle = \frac{1}{\sqrt{n}} a^\dagger |\phi_{n-1}\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |\phi_0\rangle$$

– $a^\dagger$ acts as an <u>excitation operator</u>, a <u>raising operator</u>; it increases the energy by ω_0

$$[N, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a$$

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] = a^\dagger$$

The eigenbasis we want satisfies:

$$N|\phi_\nu\rangle = \nu|\phi_\nu\rangle$$

$$H|\phi_\nu\rangle = (\nu+\tfrac{1}{2})\omega_0|\phi_\nu\rangle$$

$$Na|\phi_\nu\rangle = \{aN+[N,a]\}|\phi_\nu\rangle$$

$$= \nu a|\phi_\nu\rangle - a|\phi_\nu\rangle$$

$$= (\nu-1)\,a|\phi_\nu\rangle$$

$$\Rightarrow |\phi_{\nu-1}\rangle = C_\nu a|\phi_\nu\rangle$$

$$Na^\dagger|\phi_\nu\rangle = (\nu+1)\,a^\dagger|\phi_\nu\rangle$$

$$|\phi_{\nu+1}\rangle = C'_\nu a^\dagger|\phi_\nu\rangle$$

$$\langle\phi_\nu|H|\phi_\nu\rangle > 0 \quad \Big\}\ \text{already proven}$$

$$\boxed{\nu > \tfrac{1}{2}}$$

$$\Rightarrow a|\phi_0\rangle = 0$$

must end. tower $a|\phi_\nu\rangle$

Spectrum must be:

$$N|\phi_\nu\rangle = \nu|\phi_\nu\rangle$$

$\nu = 0, 1, 2, 3, \ldots \longleftarrow$ must be integers

can we try $\frac{1}{2} > \nu_0 > 0$; $a|\phi_{\nu_0}\rangle = 0 \Rightarrow N|\phi_{\nu_0}\rangle = 0 \Rightarrow \nu_0 = 0$

Let's find $|\phi_0\rangle$

$$a|\phi_0\rangle = \frac{1}{\sqrt{2}}\left(\beta\hat{X} + \frac{i}{\beta}\hat{P}\right)|\phi_0\rangle = 0$$

$\xrightarrow[\text{position}]{}$ $\left(\beta^2 x + \frac{d}{dx}\right)\phi_0(x) = 0$ } only one solution

ground state is gaussian

$$\boxed{\phi_0(x) = \left(\frac{\beta^2}{\pi}\right)^{1/4} \exp\left\{-\frac{1}{2}\beta^2 x^2\right\}}$$

$$\beta = \sqrt{Km}$$

already seen $a^{+}|\phi_n\rangle = C_n|\phi_{n+1}\rangle$

if $\langle\phi_n|\phi_n\rangle = 1$, then

$$\langle\phi_{n+1}|\phi_{n+1}\rangle = \frac{1}{|C|^2}\langle\phi_n|aa^{+}|\phi_n\rangle$$

$$= \frac{1}{|C|^2}\langle\phi_n|\left(N + [a,a^{+}]\right)|\phi_n\rangle$$

$$= \frac{n+1}{|C|^2}\langle\phi_n|\phi_n\rangle = \frac{n+1}{|C|^2}$$

$$\Rightarrow C_n = \sqrt{n+1} \quad ; \quad |\phi_{n+1}\rangle = \frac{1}{\sqrt{n+1}}a^{+}|\phi_n\rangle$$

by induction: $\boxed{|\phi_n\rangle = \frac{1}{\sqrt{n!}}(a^{+})^n|\phi_0\rangle}$

in position representation:

$$\boxed{\phi_n(x) = \frac{1}{\sqrt{2^n n!\,\beta^n}}\left(\beta^2 x - \frac{d}{dx}\right)^n\left(\frac{\beta^2}{\pi}\right)^{1/4}\exp\left[-\tfrac{1}{2}\beta^2 x^2\right]}$$

$$\langle\phi_m|\phi_n\rangle = \delta_{mn}$$

$$\sum_n |\phi_n\rangle\langle\phi_n| = \hat{\mathbb{1}}$$

These are Hermite polynomials.

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}$$

$$H_0(z) = 1$$

$$H_1(z) = 2z$$

$$H_2(z) = 4z^2 - 2$$

$$H_3(z) = 8z^3 - 12z$$

$$\boxed{\phi_n(x) = \left(\frac{\beta^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-\beta^2 x^2/2} H_n(\beta x)}$$