Spherical Harmonics

in position representation:

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

We want solutions for:

$$L_z\, f(\theta,\phi) = m\, f(\theta,\phi)$$

$$L^2\, f(\theta,\phi) = \ell(\ell+1)\, f(\theta,\phi)$$

$$f(\theta,\phi) = F(\theta)\, G(\phi)$$

$$-i\hbar \frac{\partial}{\partial\phi} G(\phi) = m\, G(\phi)$$

$$G(\phi) = e^{im\phi}$$

$$G(\phi+2\pi) = e^{im(\phi+2\pi)} = e^{im\phi}$$

$$e^{im2\pi} = 1 \quad \Rightarrow \quad m = 0, \pm 1, \pm 2, \ldots$$

To find $F(\theta)$, use the fact that when $m = \ell$: $L_+ F(\theta) e^{i\ell\phi} = 0$.

$$L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right)$$

$$L_- = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right)$$

$$L_+ F(\theta) e^{i\ell\phi} = e^{i(\ell+1)\phi}\left(\frac{\partial}{\partial\theta} - \ell\cot\theta\right)F(\theta) = 0$$

solution : $F(\theta) = N(\sin\theta)^\ell$ $\left\{ \begin{array}{l} \text{need} \\ F(\theta) = 0 \end{array} \right.$

For a given ℓ, m these solutions are called:

$$F(\theta)\,G(\phi) = Y_\ell^m(\theta,\phi)$$

$$Y_\ell^\ell(\theta,\phi) = N(\sin\theta)^\ell e^{i\ell\phi}$$

$$L_- Y_\ell^\ell(\theta,\phi) = \sqrt{\ell(\ell+1) - \ell(\ell-1)}\; Y_\ell^{\ell-1}(\theta,\phi)$$

etc.

ℓ	m	$Y_\ell^m(\theta, \phi)$
0	0	$Y_0^0 = \sqrt{\frac{1}{4\pi}}$
1	1	$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$
1	0	$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$
1	-1	$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}$

phase convention: $Y_\ell^{-m}(\theta, 0) = (-1)^m Y_\ell^m(\theta, 0)$

normalization

$$\int d^3r \longrightarrow \int_0^\infty r^2 \, dr \underbrace{\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi}_{\int d\Omega}$$

$$\int d\Omega \; Y_\ell^{m*}(\theta, \phi) \, Y_{\ell'}^{m'}(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'}$$