Harmonic Oscillator + Hermite Polynomials

$$\omega_o = \sqrt{\frac{K}{m}} \; ; \quad \frac{1}{2} K x^2 = \frac{1}{2} m \omega_o^2 x^2$$

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega_o^2 x^2 \right] \psi_n(x) = E_n \psi_n(x)$$

$$\text{Let } \beta = \sqrt{\frac{m\omega_o}{\hbar}} \; ; \quad E_n = \left(n + \frac{1}{2}\right) \hbar \omega_o$$

$$\psi_n(x) = \left(\frac{\beta^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\beta x) \, e^{-\beta^2 x^2/2}$$

$H_n(z)$ - Hermite polynomials

$$H_n(z) = \left(2z - \frac{d}{dz}\right)^n 1$$

$$H_o(z) = 1$$

$$H_1(z) = 2z$$

$$H_2(z) = 4z^2 - 2$$

$$H_3(z) = 8z^3 - 12z$$

Harmonic Oscillator $\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}K\hat{X}^2$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right)$$

$$\langle x|\psi_1\rangle = \left(\frac{\beta^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}} (2\beta x)\, e^{-\beta^2 x^2/2}$$

$$\langle x|\psi_2\rangle = \left(\frac{\beta^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{8}} (4\beta^2 x^2 - 2)\, e^{-\beta^2 x^2/2}$$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\, |\psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}} \Big\{ \exp\left[-i\left(\tfrac{3}{2}\omega\right)t\right] |\psi_1\rangle$$

$$+ \exp\left[-i\left(\tfrac{5}{2}\omega\right)t\right] |\psi_2\rangle \Big\}$$

$\langle p|\psi(t)\rangle$ requires $\langle p|\psi_1\rangle$ and $\langle p|\psi_2\rangle$.

$$\langle p|\psi_1\rangle = \int dx\, \langle p|x\rangle\langle x|\psi_1\rangle$$

$$= \int dx\, \frac{e^{-ipx}}{\sqrt{2\pi}} \left(\frac{\beta^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}} (2\beta x)\, e^{-\beta^2 x^2/2}$$

This leads to :

$$\int_{-\infty}^{\infty} dx \quad x \exp\left[-\alpha x^2 - ipx\right]$$

$$= \int_{-\infty}^{\infty} dx \quad x \exp\left[-\alpha\left(x + \frac{ip}{2\alpha}\right)^2 - \frac{p^2}{4\alpha}\right]$$

$$= e^{-p^2/4\alpha} \int_{-\infty}^{\infty} dy \left(\cancelto{0}{y} - \frac{ip}{2\alpha}\right) \exp\left[-\alpha y^2\right]$$

$$= \left(-\frac{ip}{2\alpha}\right) e^{-p^2/4\alpha} \sqrt{\frac{\pi}{\alpha}}$$

$$\langle p | \psi_1 \rangle = \frac{\beta}{\sqrt{\pi}} \left(\frac{\beta^2}{\pi}\right)^{1/4} \sqrt{\frac{2\pi}{\beta^2}} \left(-\frac{ip}{\beta^2}\right) e^{-p^2/2\beta^2}$$

$$\boxed{\langle p | \psi_1 \rangle = -i \left(\frac{4\beta^2}{\pi}\right)^{1/4} \frac{p}{\beta^2} e^{-p^2/2\beta^2}}$$