Eigenstates of L_x for $\ell = 1$; $\hbar = 1$

$$L^2 |\ell, m\rangle = \ell(\ell+1) |\ell, m\rangle$$
$$L_z |\ell, m\rangle = m |\ell, m\rangle$$

} no subscript on m for L_z eigenstates

We want $|\ell, m_x\rangle$: $L_x |\ell, m_x\rangle = m_x |\ell, m_x\rangle$

Any basis can be expanded in terms of any other basis:

$$|\ell, m_x\rangle = \sum_{m=-\ell}^{\ell} c_{\ell m}^{(m_x)} |\ell, m\rangle$$

$$\Rightarrow \quad |1, m_x\rangle = \sum_{m=-1}^{1} c_{1m}^{(m_x)} |1, m\rangle$$

Note: $L_+ |1,1\rangle = 0 \qquad L_+ |1,0\rangle = \sqrt{2}\,|1,1\rangle$

$$L_+ |1,-1\rangle = \sqrt{2}\,|1,0\rangle$$

$$L_- |1,1\rangle = \sqrt{2}\,|1,0\rangle \qquad L_- |1,0\rangle = \sqrt{2}\,|1,-1\rangle$$

$$L_- |1,-1\rangle = 0$$

$$L_x = \frac{1}{2}\left(L_+ + L_-\right)$$

solve for $|1, 1_x\rangle = c_{-1}|1,-1\rangle + c_0|1,0\rangle + c_1|1,1\rangle$

$$L_x|1,1_x\rangle = \tfrac{1}{2}(L_+ + L_-)\left[c_{-1}|1,-1\rangle + c_0|1,0\rangle + c_1|1,1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[c_{-1}|1,0\rangle + c_0|1,1\rangle + c_0|1,-1\rangle + c_1|1,0\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[c_0|1,-1\rangle + (c_{-1}+c_1)|1,0\rangle + c_0|1,1\rangle\right]$$

$$= |1,1_x\rangle = c_{-1}|1,-1\rangle + c_0|1,0\rangle + c_1|1,1\rangle$$

$$\Rightarrow \quad \frac{c_0}{\sqrt{2}} = c_{-1}, \quad \frac{c_{-1}+c_1}{\sqrt{2}} = c_0, \quad \frac{c_0}{\sqrt{2}} = c_1$$

These determine the c's to within an overall multiplicative constant. A solution is: $c_1 = c_{-1} = \frac{1}{2}$, $c_0 = \frac{1}{\sqrt{2}}$.

We do the same thing for $|1,0_x\rangle$ and $|1,-1_x\rangle$:

$$|1,1_x\rangle = \frac{1}{2}\left\{|1,1\rangle + \sqrt{2}\,|1,0\rangle + |1,-1\rangle\right\}$$

$$|1,0_x\rangle = \frac{1}{\sqrt{2}}\left\{|1,1\rangle - |1,-1\rangle\right\}$$

$$|1,-1_x\rangle = \frac{1}{2}\left\{|1,1\rangle - \sqrt{2}\,|1,0\rangle + |1,-1\rangle\right\}$$

These are <u>not</u> unique. For example,
$|1, 0_x\rangle = \frac{1}{2}\left\{|1,-1\rangle - |1,1\rangle\right\}$ would work
just as well. With these choices some
simple algebra then yields:

$$|1,1\rangle = \frac{1}{2}\left\{|1,1_x\rangle + \sqrt{2}\,|1,0_x\rangle + |1,-1_x\rangle\right\}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left\{|1,1_x\rangle - |1,-1_x\rangle\right\}$$

$$|1,-1\rangle = \frac{1}{2}\left\{|1,1_x\rangle - \sqrt{2}\,|1,0_x\rangle + |1,-1_x\rangle\right\}$$