PHYSICS 632 **Winter 2005** **Jan. 27 midterm**

Name: Solutions

30 pts

1) A particle in a sphere, $r < a$, is in the state:
$\psi(\mathbf{r}, t=0) = \frac{1}{\sqrt{2}}[\phi_1(\mathbf{r}) - \phi_2(\mathbf{r})]$,
where ϕ_n are the energy eigenstates, with energies E_n, ordered from lowest eigenvalue to highest. The first state is the $l = m = 0$ ground state. (a) What is the lowest energy that can be measured? Assume mass M and give the answer in terms of M and a. (b) What is $\psi(\vec{x}, t)$? (c) What is $< \hat{R} > (t)$, where $< r|\hat{R}|f >= rf(r)$. Write your answer in terms of well-defined radial integrals.

$$\psi(\vec{r}, t=0) = \frac{1}{\sqrt{2}}\Big\{ A_{100}\, j_0(k_{01} r)\, Y_0^0(\theta,\phi) + A_{11m}\, j_1(k_{11} r)\, Y_1^m(\theta,\phi) \Big\}$$

The second energy is $l = 1$ and $m = 0, \pm 1$ all have this same energy.

$$E_{10} = \frac{\hbar^2 k_{01}^2}{2M} = \frac{\hbar^2 \pi^2}{2Ma^2}, \qquad E_{11} = \frac{\hbar^2 k_{11}^2}{2M} = \frac{\hbar^2 (4.49)^2}{2Ma^2}$$

(a) E_{10} is the lowest energy that can be found. $\boxed{E_{10} = \frac{\hbar^2 \pi^2}{2Ma^2}}$

(b) $$\psi(\vec{r}, t) = \frac{1}{\sqrt{2}}\Big\{ e^{-i\left(\frac{E_{10}}{\hbar}\right)t} A_{100}\, j_0(k_{01} r)\, Y_0^0(\theta,\phi) + e^{-i\left(\frac{E_{11}}{\hbar}\right)t} A_{11m}\, j_1(k_{11} r)\, Y_1^m(\theta,\phi) \Big\}$$

(c) angular integrals kill cross terms, + with no cross terms there is no time dependence

$$\langle \hat{R} \rangle = \frac{1}{2}\int_0^a r^3 dr \Big\{ |A_{100}|^2 \big(j_0(k_{01} r)\big)^2 + |A_{11m}|^2 \big(j_1(k_{11} r)\big)^2 \Big\}$$

The extra factor of r from the operator.

1

20 pts

2) $\hat{H} = \frac{\hat{L}^2}{2I} + \omega_0 \hat{L}_z$. What are the eigenstates and eigenvalues of $\hat{H}$? At $t = 0$ the particle is in the lowest energy eigenstates with $l = 1$. What is $< \hat{L}_x > (t)$.

- eigenstates are $|\ell m\rangle$, where $\langle \theta,\phi | \ell m\rangle = Y_\ell^m(\theta,\phi)$

$$L^2 |\ell m\rangle = \ell(\ell+1)\hbar^2 |\ell m\rangle, \quad L_z |\ell m\rangle = m\hbar |\ell m\rangle$$

$$\hat{H} |\ell m\rangle = \underbrace{\left(\frac{\ell(\ell+1)\hbar^2}{2I} + m\hbar\omega_0\right)}_{\text{eigenvalues}} |\ell m\rangle$$

The state of lowest energy with $\ell = 1$ is $|1,-1\rangle$, with energy $\frac{\hbar^2}{2I} - \hbar\omega_0$. $|\psi(0)\rangle = |1,-1\rangle$,

$|\psi(t)\rangle = \exp\left[-i\left(\frac{\hbar^2}{2I} - \hbar\omega_0\right)t\right] |1,-1\rangle$, but we do not need this because there are no cross terms in:

$$\langle \hat{L}_x \rangle = \langle 1,-1 | \tfrac{1}{2}(L_+ + L_-) | 1,-1\rangle = 0$$

$\hat{L}_1 \cdot \hat{L}_2$, dot is in wrong place

$\boxed{20 \text{ pts}}$

3) (a) Compute $[\hat{L}_i, \hat{R}^2]$. (b) If $\hat{\mathbf{L}} = \hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2$, what is $[\hat{L}^2, \hat{\mathbf{L}}_1 \dot{\hat{\mathbf{L}}}_2]$? Show your work.

a) $$[L_i, \hat{R}^2] = \epsilon_{iab} [R_a P_b, R_j R_j] = \epsilon_{iab} R_a [P_b, R_j R_j]$$
$$= \epsilon_{iab} R_a \{ R_j [P_b, R_j] + [P_b, R_j] R_j \}$$
$$= \epsilon_{iab} R_a \{ -i\hbar \delta_{bj} R_j - i\hbar \delta_{bj} R_j \}$$
$$= -2i\hbar \underbrace{\epsilon_{iab}}_{\text{anti-symetric in } a \,\&\, b} \underbrace{R_a R_b}_{\text{symmetric in } a \,\&\, b} = \underline{0}$$

note this is $(\vec{R} \times \vec{R})_i = 0$

– there are many correct ways to do this calculation

b) $\hat{L} = \hat{L}_1 + \hat{L}_2$, $[L^2, \vec{L}_1 \cdot \vec{L}_2] = [L_1^2 + L_2^2 + \cancel{2\vec{L}_1 \cdot \vec{L}_2}, \vec{L}_1 \cdot \vec{L}_2]$

0 commutes with itself

$= \underline{0}$

$[L_1^2, \vec{L}_1 \cdot \vec{L}_2] = 0$ because $[L_1^2, L_{1i}] = 0$

etc.

Note: $S_x \to \pm\frac{1}{2}\hbar$; $\mathcal{P}(+\frac{1}{2}\hbar) = \left| {}_x\langle +|\psi(t)\rangle \right|^2$, $\mathcal{P}(-\frac{1}{2}\hbar) = \left| {}_x\langle -|\psi(t)\rangle \right|^2$

$\Leftrightarrow$ NEED $|\pm\rangle_x$ to compute individual probabilities.

30pts

4) For angular momentum one-half there are two eigenstates, $|+>$ and $|->$, where $J^2|\pm>= 3/4|\pm>$ and $J_z|\pm>= \pm 1/2|\pm>$. $H = \omega_0 J_z$. At $t = 0$, $|\psi(t = 0)>= \frac{1}{\sqrt{2}}(|+>+|->)$. What is $|\psi(t)>$? What are the possible measurements of J_x? What are the probabilities of each as a function of time?

$$\hat{H}|+\rangle = \frac{1}{2}\hbar\omega_0|+\rangle, \quad \hat{H}|-\rangle = -\frac{1}{2}\hbar\omega_0|-\rangle$$

$$\boxed{|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i(\frac{1}{2}\hbar\omega_0)t}|+\rangle + e^{i(\frac{1}{2}\hbar\omega_0)t}|-\rangle\right)}$$

Find eigenstates of J_x. We know the eigenvalues are $\pm\frac{1}{2}\hbar$, so we want:

Need $|\psi(t)\rangle$ in terms of: $|\pm\rangle_x$

$$|\pm\rangle_x = \alpha_\pm|+\rangle_z + \beta_\pm|-\rangle_z, \quad \text{with } |\alpha|^2+|\beta|^2=1$$

$$J_x|\pm\rangle_x = \frac{1}{2}(J_+ + J_-)|\pm\rangle_x = \frac{1}{2}\beta_\pm J_+|-\rangle_z + \frac{1}{2}\alpha_\pm J_-|+\rangle_z$$

$$= \pm\frac{1}{2}\hbar|\pm\rangle$$

$$\boxed{\begin{aligned} J_+|-\rangle &= \hbar\sqrt{\tfrac{1}{2}(\tfrac{3}{2}) + \tfrac{1}{2}(1-\tfrac{1}{2})}\,|+\rangle = \hbar|+\rangle \\ J_-|+\rangle &= \hbar|-\rangle \end{aligned}}$$

$$\frac{1}{2}\beta_\pm|+\rangle_z + \frac{1}{2}\alpha_\pm|-\rangle_z = \pm\frac{1}{2}\alpha_\pm|+\rangle_z \pm \beta_\pm|-\rangle_z$$

$$\beta_\pm = \pm\alpha_\pm \qquad \boxed{\begin{aligned} |+\rangle_x &= \tfrac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z) \\ |-\rangle_x &= \tfrac{1}{\sqrt{2}}(|+\rangle_z - |-\rangle_z) \end{aligned}}$$

$$|\psi(t)\rangle = \frac{1}{2}\left\{\exp\left[\frac{-i\hbar\omega_0}{2}t\right](|+\rangle_x + |-\rangle_x) + \exp\left[\frac{i\hbar\omega_0}{2}t\right](|+\rangle_x - |-\rangle_x)\right\}$$

$$\boxed{|\psi(t)\rangle = \cos\left(\frac{\hbar\omega_0}{2}t\right)|+\rangle_x - i\sin\left(\frac{\hbar\omega_0}{2}t\right)|-\rangle_x}$$

$\hookrightarrow \mathcal{P}(+\frac{1}{2}) = |\cos(\frac{\hbar\omega_0}{2}t)|^2$ $\qquad \hookrightarrow \mathcal{P}(-\frac{1}{2}) = |\sin(\frac{\hbar\omega_0}{2}t)|^2$