Physics 631 { AUØ6 2nd Midterm Notes

Basic Quantum Mechanics Formalism

states : $|\psi(t)\rangle \longrightarrow$ position rep. $\Psi(x,t)$

dynamics : $i\hbar \frac{d}{dt}\Psi(x,t) = \hat{H}\Psi(x,t)$; $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

normalization : $1 = \int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2 = \int_{-\infty}^{\infty} dx\, |\Psi(x,t=0)|^2$

Energy Eigenstates

$$\underbrace{H\psi_n = E_n \psi_n}$$

E_n: eigenvalue; ψ_n: eigenstate

$$\int_{-\infty}^{\infty} dx\, \psi_m(x)\psi_n(x) = \delta_{mn} \leftarrow \text{orthonormality}$$

$$\Psi(x,0) = \sum_n c_n \psi_n \; ; \; c_n = \int_{-\infty}^{\infty} dx\, \psi_n^*(x)\Psi(x,0) \leftarrow \text{completeness}$$

$$\Psi(x,t) = \sum_n c_n e^{-iE_nt/\hbar}\psi_n(x) \; ; \; 1 = \sum_n |c_n|^2$$

probabity to measure E_n : $P(E_n) = |c_n|^2$

probabity position : $a<x<b$ $\quad P_{ab} = \int_a^b dx\, |\Psi(x,t)|^2$

average position : $\langle \hat{x} \rangle = \int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2\, x$

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2\, x^2 \text{ , etc.}$$

average momentum : $\langle \hat{p} \rangle = \int_{-\infty}^{\infty} dx\, \Psi^*(x,t)\left(-i\hbar\frac{d}{dx}\right)\Psi(x,t)$

2nd Midterm Notes

Square Well $\quad H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{sq.}(x) \qquad V_{sq.}(x) = \begin{cases} 0, & \text{inside} \\ \infty, & \text{outside} \end{cases}$

$\Psi(x,t) = 0$ outside well and at both edges

For $E = \frac{\hbar^2 k^2}{2m}$, $\psi(x) = C\sin(kx) + D\cos(kx)$ inside.

Find k by solving $\psi(X_L) = \psi(X_R) = 0$, where X_L is the left edge and X_R is the right edge.

If $X_L = 0$ and $X_R = a$,

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), \quad E_n = n^2 E_1, \quad E_1 = \frac{\hbar^2\pi^2}{2ma^2}$$

$$n = 1, 2, 3, \ldots$$

Harmonic Oscillator $\quad H = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 = \left(a_+a_- + \frac{1}{2}\right)\hbar\omega$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\hat{P} + m\omega\hat{X}\right)$$

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\left(a_+ + a_-\right) \quad ; \quad \hat{P} = i\sqrt{\frac{\hbar m\omega}{2}}\left(a_+ - a_-\right)$$

$$\left[\hat{X}, \hat{P}\right] = i\hbar \Rightarrow \left[a_-, a_+\right] = 1$$

$$\psi_n = \frac{1}{\sqrt{n!}}\left(a_+\right)^n\psi_0 \quad ; \quad E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$\int_{-\infty}^{\infty} dx\, \psi_m(x)\psi_n(x) = \delta_{mn}$$

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases} \qquad \frac{d}{dx}\Theta(x) = \delta(x)$$

$$\int_{-\infty}^{\infty} dx\ \delta(x)\,f(x) = f(0) \quad ; \quad \delta(cx) = \frac{1}{|c|}\delta(x)$$

For $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\,\delta(x)$ (attractive δ-function)

boundary conditions are :

$$\psi_{left}(0) = \psi_{right}(0)$$

$$\left.\frac{d\psi_{rt}}{dx}\right|_{x=0} - \left.\frac{d\psi_{left}}{dx}\right|_{x=0} = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

Some Integrals

$$\int_0^a dx\ \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right) = \frac{a}{2}\delta_{mn}$$

$$m \neq n \quad \int_0^a dx\ \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right) x = \frac{2\left(-1+(-1)^{m+n}\right) mn\, a^2}{(m^2-n^2)^2\pi^2}$$

note error in handout

$$\xrightarrow[m=n]{} \frac{a^2}{4}$$

$$m \neq n \quad \int_0^a dx\ \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right) x^2 = \frac{4(-1)^{m+n}\, mn\, a^3}{(m^2-n^2)^2\ \pi^2}$$

$$\xrightarrow[m=n]{} \frac{a^3}{12}\left(2 - \frac{3}{n^2\pi^2}\right)$$

$$m \neq n \quad \int_0^{a/2} dx\ \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$

$$= \frac{a\, n \cos\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right) - a\, m\cos\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)}{(m^2-n^2)\ \pi^{\cancel{2}}}$$

note another error

$$\xrightarrow[m=n]{} \frac{a}{4}$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

$$\cos(a)\sin(b) = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right]$$