PHYSICS 631 **Autumn 2005** **Midterm #3**

Name: Solutions

Do NOT simply write an answer. Give a calculation and/or reasoning that supports your answer. Either copy a cleaned-up answer to the test sheet or organize what you write in the text book. If there are several different calculations near your answers, you are responsible for all errors.

1) A particle of mass m is bound in an infinite square well so that it is free for $0 < x < a$ and unable to leave this region. At time $t = 0$, it is in a simple state localized around the middle of the box at $x = a/2$. At $t = 0$, $< x|\psi(0) >= N\theta(x - a/4)\theta(3a/4 - x)$. In words, it is equal to N from $x = a/4$ to $x = 3a/4$ and it is zero everywhere else. (a) What is $< x|\psi(t) >$ for later times? Evaluate any integrals you encounter and fix N so that the state is normalized. (b) If energy is measured at time t, what are the possible results and what is the probability for each result?

$$\langle\psi(0)|\psi(0)\rangle = N^2\int_{a/4}^{3a/4} dx = N^2\left(\frac{a}{2}\right) \Rightarrow N = \sqrt{\frac{2}{a}}$$

$$(a)\ \langle x|\psi(t)\rangle = \langle x|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \sum_n \langle x|\phi_n\rangle\langle\phi_n|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$$

$$= \sum_n \langle x|\phi_n\rangle\langle\phi_n|e^{-iE_nt/\hbar}|\psi(0)\rangle = \sum_n e^{-iE_nt/\hbar}\langle x|\phi_n\rangle\langle\phi_n|\psi(0)\rangle$$

$$\langle\phi_n|\psi(0)\rangle = N\int_{a/4}^{3a/4} dx\, \phi_n(x) = \frac{2}{a}\int_{a/4}^{3a/4} dx \sin\left(\frac{n\pi x}{a}\right) = \frac{2}{n\pi}\left[-\cos\left(\frac{n\pi x}{a}\right)\right]\Bigg|_{a/4}^{3a/4}$$

$$= \frac{2}{n\pi}\left[-\cos\left(\frac{3n\pi}{4}\right) + \cos\left(\frac{n\pi}{4}\right)\right] = \frac{\sqrt{2}}{n\pi}\left(1-(-1)^n\right)$$

$$\langle x|\psi(t)\rangle = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n\pi}\left(1-(-1)^n\right)\exp\left(-i\frac{\hbar\pi^2}{2ma^2}n^2t\right)\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

$$(b)\ E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2,\quad n = 1, 2, \ldots$$

$$P(E_n) = |\langle\phi_n|\psi(t)\rangle|^2 = |\langle\phi_n|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle|^2$$

$$= |e^{-iE_nt/\hbar}\langle\phi_n|\psi(0)\rangle|^2 = |\langle\phi_n|\psi(0)\rangle|^2$$

$$= \frac{2}{n^2\pi^2}\left(1-(-1)^n\right)^2$$

2) A particle of mass m is free, $H = P^2/(2m)$. At $t = 0$ the particle is in the initial state $< x|\psi(0) >= N\theta(x)\theta(a-x)$, a constant from $x = 0$ to $x = a$. (a) What is $< x|\psi(t) >$ for later times? Evaluate any integrals you encounter and fix N so that the state is normalized. (b) If momentum is measured at time t, what is the probability per unit of momentum for finding momentum p. (c) If energy is measured at time t, what are the possible results and what is the probability for 'each' result?

$$\langle \psi(0)|\psi(0)\rangle = N^2\int_0^a dx = N^2 a \Rightarrow N = \sqrt{\frac{1}{a}}$$

$$(a)\ \langle x|\psi(t)\rangle = \langle x|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \int_{-\infty}^{\infty} dk\, \langle x|k\rangle\langle k|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$$

$$= \int_{-\infty}^{\infty} dk\, \frac{e^{ikx}}{\sqrt{2\pi}} \exp\left(-i\frac{\hbar k^2}{2m}t\right)\langle k|\psi(0)\rangle$$

$$\langle k|\psi(0)\rangle = \int_{-\infty}^{\infty} dx\, \langle k|x\rangle\langle x|\psi(0)\rangle = \sqrt{\frac{1}{a}}\int_0^a dx\, \frac{e^{-ikx}}{\sqrt{2\pi}}$$

$$= \sqrt{\frac{1}{2\pi a}}\left.\frac{e^{-ikx}}{-ik}\right|_0^a = \sqrt{\frac{1}{2\pi a}}\,\frac{1-e^{-ika}}{ik}$$

$$\langle x|\psi(t)\rangle = \frac{-i}{2\pi\sqrt{a}}\int_{-\infty}^{\infty} dk\, \exp\left(ikx - i\frac{\hbar t}{2m}k^2\right)\left(\frac{1-e^{-ika}}{k}\right)$$

$$(b)\ P(p=\hbar k) = |\langle k|\psi(t)\rangle|^2 = |\langle k|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle|^2$$

$$= \left|\langle k|e^{-i\frac{\hbar k^2}{2m}t}|\psi(0)\rangle\right|^2 = |\langle k|\psi(0)\rangle|^2$$

$$= \frac{1}{2\pi a}\left(\frac{1-e^{-ika}}{ik}\right)\left(\frac{1-e^{ika}}{-ik}\right) = \frac{1}{2\pi a}\left(\frac{2-2\cos(ka)}{k^2}\right)$$

$$= \frac{1-\cos(ka)}{\pi a k^2}$$

$$(c)\ P\left(E=\frac{\hbar^2k^2}{2m}\right) = |\langle k|\psi(t)\rangle|^2 = \frac{1-\cos(ka)}{\pi a k^2}$$

3) A particle of mass m is bound by a harmonic oscillator potential. See the handout for all details that you need. (a) In this part, $< x|psi(0) >= N\exp(-x^2/a^2)$, which has a different width from the ground state. What is $< x|\psi(t) >$? Your answer can include integrals that are not evaluated, but everything in your expression should be completely defined. Choose N to normalize the state. (b) In this part $|\psi(0) >= 1/\sqrt{2}\Big(|\psi_0 > +|\psi_1 >\Big)$. Here, $|\psi_0 >$ is the ground state and $|\psi_1 >$ is the first excited state. What is $< p|\psi(t) >$? Evaluate any integrals you encounter.

$$(a)\ \langle\psi(0)|\psi(0)\rangle = N^2\int_{-\infty}^{\infty} dx\, e^{-2x^2/a^2} = N^2\sqrt{\frac{\pi a^2}{2}}$$

$$\Rightarrow N = \left(\frac{2}{\pi a^2}\right)^{1/4}$$

$$\langle x|\psi(t)\rangle = \langle x|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$$

$$= \sum_n \langle x|\psi_n\rangle\langle\psi_n|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$$

$$= \sum_n \exp\left(-i\left(n+\tfrac{1}{2}\right)\omega t\right)\psi_n(x)\langle\psi_n|\psi(0)\rangle$$

$$\langle\psi_n|\psi(0)\rangle = \int_{-\infty}^{\infty} dx \left(\frac{\beta^2}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\beta x)\, e^{-\beta^2x^2/2}\left(\frac{2}{\pi a^2}\right)^{1/4} e^{-x^2/a^2}$$

$$(b)\ \langle p|\psi(t)\rangle = \langle p|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}}\langle p|\left\{e^{-iE_0t/\hbar}|\psi_0\rangle + e^{-iE_1t/\hbar}|\psi_1\rangle\right\}$$

$$\langle p|\psi_0\rangle = \int_{-\infty}^{\infty} dx\,\frac{e^{-ipx}}{\sqrt{2\pi}}\left(\frac{\beta^2}{\pi}\right)^{1/4} e^{-\beta^2x^2/2} \qquad \left\{\begin{array}{l}\text{exponent}\\ = -\frac{\beta^2}{2}\left(x+\frac{ip}{\beta^2}\right)^2 - \frac{p^2}{2\beta^2}\end{array}\right.$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{\beta^2}{\pi}\right)^{1/4}\sqrt{\frac{2\pi}{\beta^2}}\, e^{-p^2/2\beta^2} = \left(\frac{1}{\pi\beta^2}\right)^{1/4} e^{-p^2/2\beta^2}$$

$$\langle p|\psi_1\rangle = \int_{-\infty}^{\infty} dx\,\frac{e^{-ipx}}{\sqrt{2\pi}}\left(\frac{\beta^2}{\pi}\right)^{1/4}\frac{1}{\sqrt{2}}(2\beta x)e^{-\beta^2x^2/2} = \frac{\beta}{\sqrt{\pi}}\left(\frac{\beta^2}{\pi}\right)^{1/4}\int_{-\infty}^{\infty} dy\left(\cancel{y}^{\,\text{odd}} - \frac{ip}{\beta^2}\right)\cdots$$

$$= \frac{\beta}{\sqrt{\pi}}\left(\frac{\beta^2}{\pi}\right)^{1/4}\left(-\frac{ip}{\beta^2}\right)\sqrt{\frac{2\pi}{\beta^2}}\, e^{-p^2/2\beta^2} = \sqrt{2}\left(\frac{\beta^2}{\pi}\right)^{1/4}\left(\frac{-ip}{\beta^2}\right)e^{-p^2/2\beta^2}$$