PHYSICS 631 **Autumn 2004** **Midterm #2a**

Name: Solutions

1) (a) Compute $<\hat{X}\hat{P}>$ and $<\hat{P}\hat{X}>$ for a particle in harmonic oscillator state $|\phi_n>$. Define any constants you use in terms of m and K, where, $\hat{H} = \frac{\hat{P}^2}{2m} + \frac{K}{2}\hat{X}^2$.

(7.26) $\hat{X} = \frac{1}{\sqrt{2}\beta}(\hat{a}+\hat{a}^+)$, $\hat{P} = \frac{m\omega_0}{i}\frac{1}{\sqrt{2}\beta}(\hat{a}-\hat{a}^+)$, $\boxed{\beta = \sqrt{m\omega_0}, \quad \omega_0 = \sqrt{K/m}}$

$$\langle\phi_n|\hat{X}\hat{P}|\phi_n\rangle = \frac{m\omega_0}{2i\beta^2}\langle n|(\hat{a}+\hat{a}^+)(\hat{a}-\hat{a}^+)|n\rangle = \frac{1}{2i}\langle n|\cancel{\hat{a}^2} - \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} - \cancel{\hat{a}^{+2}}|n\rangle$$

$$= \frac{1}{2i}\langle n| -[\hat{a},\hat{a}^+]|n\rangle = -\frac{1}{2i}$$

$$\langle n|\hat{P}\hat{X}|n\rangle = \frac{1}{2i}\langle n|(\hat{a}-\hat{a}^+)(\hat{a}+\hat{a}^+)|n\rangle = \frac{1}{2i}\langle n|[\hat{a},\hat{a}^+]|n\rangle$$

$$= \frac{1}{2i}$$

Note: $\langle n|[\hat{P},\hat{X}]|n\rangle = \frac{1}{i}$

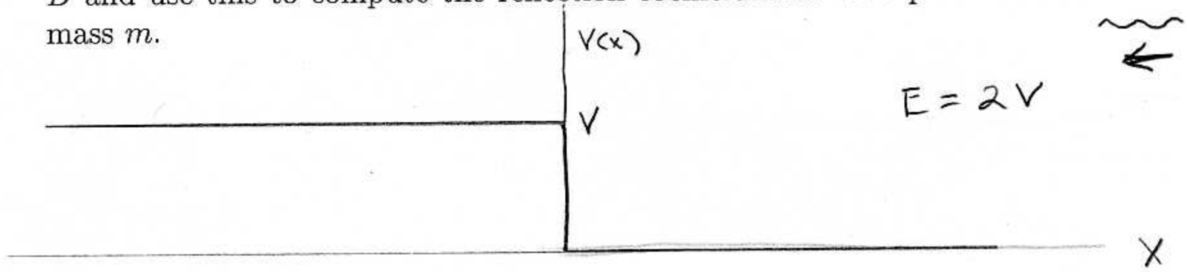
(b) Compute $<\hat{X}\hat{P}>$ for a state that is $|\psi(t=0)> = \frac{1}{\sqrt{2}}|\phi_0> + \frac{1}{\sqrt{2}}|\phi_1>$ at time $t=0$. This is a function of time. The algebraic method is the easiest way to do this calculation.

Time dependence is trivial. $\hat{X}\hat{P} \propto -\hat{a}^2 - [\hat{a},\hat{a}^+] - \hat{a}^{+2}$, and $\langle\phi_0|\hat{X}\hat{P}|\phi_1\rangle = 0$, so no terms of the form $e^{-i(\omega_1-\omega_0)t}$ survive. $\langle\phi_0|\hat{X}\hat{P}|\phi_0\rangle$ and $\langle\phi_1|\hat{X}\hat{P}|\phi_1\rangle$ are constants in time. We have

$$\langle\hat{X}\hat{P}\rangle = \frac{1}{2}\langle\phi_0|\hat{X}\hat{P}|\phi_0\rangle + \frac{1}{2}\langle\phi_1|\hat{X}\hat{P}|\phi_1\rangle$$

$$= -\frac{1}{2i} \text{ from above.}$$

2) Particles are shot at the step potential shown below from the right, so that the wavefunction for $x > 0$ is $e^{-ikx} + Be^{ikx}$. Assume $E = 2V$, compute B and use this to compute the reflection coefficient R. The particles have mass m.



$$\phi(x) = \begin{cases} e^{-ikx} + Be^{ikx} &, x \geq 0 \\ Ce^{-ik'x} &, x \leq 0 \end{cases}$$

ϕ & ϕ' are continuous at $x = 0$

$$\phi(0) = 1 + B = C$$

$$\phi'(0) = -ik(1-B) = -ik'C$$

$$\frac{k}{k'}(1-B) = C$$

$E = \frac{k^2}{2m}$; $E - V = \frac{k'^2}{2m}$; $E = 2V$

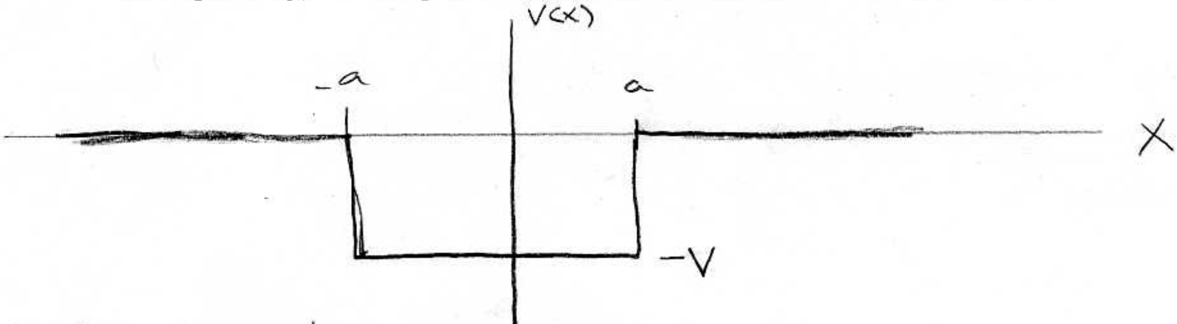
$$\Rightarrow k = \sqrt{2}\, k'$$

subtract $\rightarrow$ $1 + B - \sqrt{2}(1-B) = 0$

$$\boxed{B = \frac{\sqrt{2}-1}{\sqrt{2}+1}}$$

$$\boxed{R = |B|^2 = \frac{3-2\sqrt{2}}{3+2\sqrt{2}}}$$

3) The finite square well potential shown below has only one bound state. Derive the boundary conditions for the ground state at $x = a$. If I adjust V and a separately, how deep can I make this bound state? Prove your answer.



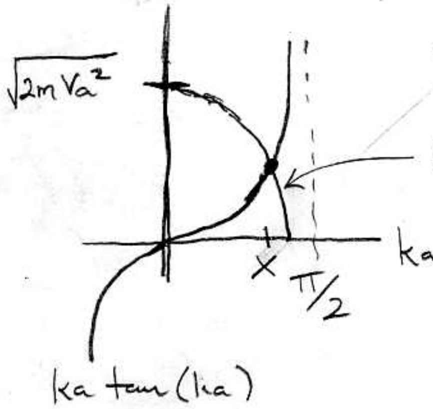
gd. state symmetric;

$\phi(x) = Ae^{-K|x|}$ $[|x| \geq a]$; $\phi(x) = B\cos(kx)$ $[|x| \leq a]$

$$E = -\frac{K^2}{2m} = \frac{k^2}{2m} - V \; ; \quad \boxed{Ka = \sqrt{2mVa^2 - k^2a^2}}$$

bd. conds.: $\phi(a) = B\cos(ka) = Ae^{-Ka}$; $\phi'(a) = -Bk\sin(ka) = -KAe^{-Ka}$

divide → $ka\cot(ka) = Ka = \sqrt{2mVa^2 - k^2a^2}$



Ka; intercept Ka=0 at $ka = \sqrt{2mVa^2}$

only one bd. state ⇒ $\sqrt{2mVa^2} < \frac{\pi}{2}$

Fix: $2mVa^2 = \xi^2$, and vary V+a together

For $\xi^2 < (\pi/2)^2$ fixed find solution: $X\tan(X) = \sqrt{\xi^2 - X^2}$

$$E = -V + \frac{X^2}{2ma^2} = \frac{-\xi^2 + X^2}{2ma^2} \xrightarrow[\;a \to 0\;]{V = \xi^2/2ma^2} -\infty$$

by letting $V \to \infty$ as $a \to 0$ we can let $E \to -\infty$ and have only one bd. state