Physics 622 WI Ø8 Exam 2 Solutions

(1) $C = \frac{\partial U}{\partial T}$, $\omega = v_s|\vec{q}|$, $\int_0^{\omega_D} d\omega\, g(\omega) = 2N$

Phonons are not conserved, so we can use the partition function since the chemical potential is always zero. For each harmonic oscillator, we have n phonon energies $E_\omega = (n+\frac{1}{2})\hbar\omega$ and:

$$Z = \prod_\omega \sum_{n_\omega=0}^{\infty} e^{-(n_\omega+\frac{1}{2})\hbar\omega\beta} = \prod_\omega \left[\frac{e^{-\frac{1}{2}\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}}\right]$$

$$\ln Z = \sum_\omega \ln\left[\,''\,\right] = \sum_\omega \left\{-\tfrac{1}{2}\hbar\omega\beta - \ln\left(1-e^{-\hbar\omega\beta}\right)\right\}$$

Here $\sum_\omega \rightarrow \int d\omega\, g(\omega) = \underbrace{2}_{\text{polarization}}\frac{L^2}{(2\pi)^2}\int d^2q = \frac{L^2}{\pi}\int q\, dq = \frac{L^2}{\pi v_s^2}\int \omega\, d\omega$

$$\boxed{\ln Z = \frac{L^2}{\pi v_s^2}\int_0^{\omega_D} \omega\, d\omega \left\{-\tfrac{1}{2}\hbar\omega\beta - \ln\left(1-e^{-\hbar\omega\beta}\right)\right\}}$$

Debye frequency: $\frac{L^2}{\pi v_s^2}\int_0^{\omega_D} d\omega\, \omega = \frac{L^2}{2\pi v_s^2}\omega_D^2 = 2N$

$$\Rightarrow \boxed{\omega_D = \sqrt{\frac{4\pi v_s^2 N}{L^2}}}$$

$$U = -\frac{\partial}{\partial\beta}\ln Z = \frac{L^2\hbar}{\pi v_s^2}\int_0^{\omega_D} d\omega \left[\frac{\omega^2 e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}} + \frac{1}{2}\omega^2\right]$$

$$= \frac{L^2\hbar}{\pi v_s^2}\left\{\int_0^{\omega_D} d\omega \frac{\omega^2}{e^{\hbar\omega\beta}-1} + \frac{1}{6}\omega_D^3\right\}$$

$$C = \frac{\partial U}{\partial T} = \frac{\partial \beta}{\partial T}\frac{\partial U}{\partial \beta} = \left(-\frac{1}{kT^2}\right)\frac{L^2\hbar^2}{\pi v_s^2}\int_0^{\omega_D} d\omega \frac{-\omega^3 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}$$

$$\boxed{C = \frac{L^2\hbar^2 k}{\pi v_s^2}\beta^2\int_0^{\omega_D} d\omega \frac{\omega^3 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2} = \frac{L^2 k}{\pi v_s^2}\left(\frac{1}{\hbar\beta}\right)^2\int_0^{\beta\hbar\omega_D} dx \frac{x^3 e^x}{(e^x-1)^2}}$$

(b) In the limit $\beta \to 0$ we can Taylor expand the integrand:

$$C \xrightarrow[\beta\to 0]{} \sim \frac{L^2 k}{\pi v_s^2}\left(\frac{1}{\hbar\beta}\right)^2\int_0^{\beta\hbar\omega_D} dx \frac{x^3}{x^2}$$

$$\boxed{C \xrightarrow[T\to\infty]{} \frac{L^2 k}{2\pi v_s^2}\omega_D^2 = 2Nk}$$

(2) V, T, N spin-$\frac{1}{2}$ Fermions, m (a) $N = ?$, $U = ?$

(b) $\mu(T) \xrightarrow[T\to 0]{} ?$, $U \xrightarrow[T\to 0]{} ?$; $E = \frac{\hbar^2 q^2}{2m}$

(a) $N = \int dE\, g(E) F(E)$, $U = \int dE\, g(E) F(E) E$

$$\frac{(2s+1)V}{(2\pi)^3}\int d^3q\, F(q) = \frac{2V}{\pi^2}\int q^2 dq\, F(q) \xrightarrow[\substack{dE = \frac{\hbar^2}{m} q\, dq \\ q^2 dq = \frac{m}{\hbar^2}\sqrt{\frac{2mE}{\hbar^2}}}]{} \frac{\sqrt{2}Vm^{3/2}}{\pi^2\hbar^3}\int dE \sqrt{E} F(E)$$

$$N = \frac{\sqrt{2}Vm^{3/2}}{\pi^2\hbar^3}\int_0^\infty dE\, \frac{\sqrt{E}}{e^{\beta(E-\mu)}+1} = \frac{\sqrt{2}Vm^{3/2}}{\pi^2\hbar^3}\left(\frac{1}{\beta}\right)^{3/2}\int_0^\infty dx\, \frac{\sqrt{x}}{e^{-\beta\mu}e^{x}+1}$$

$$= \frac{-\sqrt{2}Vm^{3/2}}{\pi^2\hbar^3}\left(\frac{1}{\beta}\right)^{3/2}\Gamma(3/2)\cdot Li_{3/2}\left(-e^{\beta\mu}\right)$$

$$\boxed{N = \frac{V m^{3/2}}{\sqrt{2}\pi^{3/2}\beta^{3/2}\hbar^3}\left[-Li_{3/2}(-e^{\beta\mu})\right] = \frac{2V}{\lambda_{th}^3}\left[-Li_{3/2}(-e^{\beta\mu})\right]}$$

$$\boxed{\frac{1}{\lambda_{th}^3} = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2}}$$

$$U = \frac{\sqrt{2}Vm^{3/2}}{\pi^2\hbar^3}\int_0^\infty dE\, \frac{E\sqrt{E}}{e^{\beta(E-\mu)}+1} = \frac{\sqrt{2}Vm^{3/2}}{\pi^2\hbar^3}\left(\frac{1}{\beta}\right)^{5/2}\int_0^\infty dx\, \frac{x^{3/2}}{e^{-\beta\mu}e^{x}+1}$$

$$= -\frac{\sqrt{2}Vm^{3/2}}{\pi^2\hbar^3}\left(\frac{1}{\beta}\right)^{5/2}\Gamma\left(\frac{5}{2}\right)Li_{5/2}\left(-e^{\beta\mu}\right)$$

$$\boxed{U = \frac{3}{2}NkT\,\frac{Li_{5/2}(-e^{\beta\mu})}{Li_{3/2}(-e^{\beta\mu})}}$$

(b) As $T \to 0$, we need to solve:

$$\frac{N}{2V} = c' T^{3/2} \left[-Li_{3/2}\left(-e^{\beta\mu}\right) \right]$$

This requires $e^{\beta\mu}$ to become large, so we use:

$$-Li_{3/2}(-z) \xrightarrow[z\to\infty]{} \frac{4}{3\sqrt{\pi}} \left[(\ln z)^{3/2} + \frac{\pi^2}{8} (\ln z)^{-1/2} \right] \ldots$$

$$\frac{N\lambda_{th}^3}{2V} \sim \frac{4}{3\sqrt{\pi}} \left[(\ln z)^{3/2} + \frac{\pi^2}{8} (\ln z)^{-1/2} \right]$$

You needed only the leading result but I'll also solve for the first correction:

$$\left(\frac{3\sqrt{\pi}\, N \lambda_{th}^3}{8V} \right)^{2/3} \sim \beta\mu \quad , \quad \mu(0) = \left(\frac{3\sqrt{\pi}\, N \, (2\pi\hbar^2)^{3/2}}{8V \; m^{3/2}} \right)^{2/3}$$

$$\boxed{\mu(0) = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 N}{2V} \right)^{2/3}} \quad \text{— the Fermi energy}$$

To find the next term, treat $\mu(T) - \mu(0)$ as small:

$$\frac{N\lambda_{th}^3}{2V} \sim \frac{4}{3\sqrt{\pi}} \left[\left[\beta\mu(0) \left(1 + \frac{\mu(T) - \mu(0)}{\mu(0)} \right) \right]^{3/2} + \frac{\pi^2}{8} \frac{1}{\sqrt{\beta\mu(0)}} \right]$$

equal

$$\frac{N\lambda_{th}^3}{2V} \sim \frac{4}{3\sqrt{\pi}}\left\{\left[\beta\mu(0)\right]^{3/2} + \frac{3}{2}\left[\beta\mu(0)\right]^{3/2}\left[\frac{\mu(T)-\mu(0)}{\mu(0)}\right] + \frac{\pi^2}{8}\frac{1}{\sqrt{\beta\mu(0)}}\right\}$$

these terms must cancel

$$\frac{\mu(T)-\mu(0)}{\mu(0)} = -\frac{\pi^2}{12\left[\beta\mu(0)\right]^2}$$

$$\boxed{\mu(T) = \mu(0) - \frac{\pi^2}{12}\left[\frac{k_B T}{\mu(0)}\right]^2 + \cdots}$$

To find the low temperature limit for U, we need $Li_{5/2}(-e^{\beta\mu}) / Li_{3/2}(-e^{\beta\mu})$ for $\mu \to \mu(0)$ and $\beta \to \infty$.

The notes gave $-Li_{3/2}(-z) \xrightarrow[z\to\infty]{} \frac{4}{3\sqrt{\pi}}(\ln z)^{3/2} + \cdots$. Use

$$Li_n(z) = \sum_{k=0}^{\infty}\frac{z^k}{k^n} \Rightarrow \frac{d}{dz}Li_n = \sum\frac{kz^{k-1}}{k^n} = \frac{1}{z}\sum\frac{z^k}{k^{n-1}} = \frac{1}{z}Li_{n-1}(z)$$

$$z\frac{d}{dz}Li_{5/2}(z) = Li_{3/2}(z) \xrightarrow[z\to-\infty]{} -\frac{4}{3\sqrt{\pi}}\left(\ln(-z)\right)^{3/2}$$

$$\Rightarrow Li_{5/2}(-z) \xrightarrow[z\to\infty]{} -\frac{8}{15\sqrt{\pi}}\left(\ln z\right)^{5/2}$$

$$U \xrightarrow[T\to 0]{} \frac{3}{2}NkT\left[\frac{8}{15}\cdot\frac{3}{4}(\beta\mu)\right] \longrightarrow \frac{3}{5}N\mu(0)$$

(3) V, T, N spin-zero bosons, m. (a) $N = ?$, $U = ?$

(b) $\mu(T)$ for $T \to 0$ $\to$ $U \sim ?$; $E = \frac{\hbar^2 q^2}{2m}$

(a) $N = \int dE\, g(E) f(E)$, $U = \int dE\, g(E) f(E) E$

$$\frac{V}{(2\pi)^3} \int d^3q\, F(q) = \frac{V}{2\pi^2} \int q^2 dq\, F(q) \xrightarrow[dE = \frac{\hbar^2}{m} q\,dq]{} \frac{V m^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int \sqrt{E}\, dE\, F(E)$$

$$q^2 dq = \frac{m}{\hbar^2} \sqrt{\frac{2mE}{\hbar^2}}\, dE$$

$$N = \frac{V m^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^\infty dE \frac{\sqrt{E}}{e^{\beta(E-\mu)} - 1} \xrightarrow[\substack{x = \beta E \\ \sqrt{E}dE = \frac{\sqrt{x}\,dx}{\beta^{3/2}}}]{} \frac{V}{\sqrt{2}\pi^2 \hbar^3} \left(\frac{m}{\beta}\right)^{3/2} \times \Gamma\left(\tfrac{3}{2}\right) Li_{3/2}\left(e^{\beta\mu}\right)$$

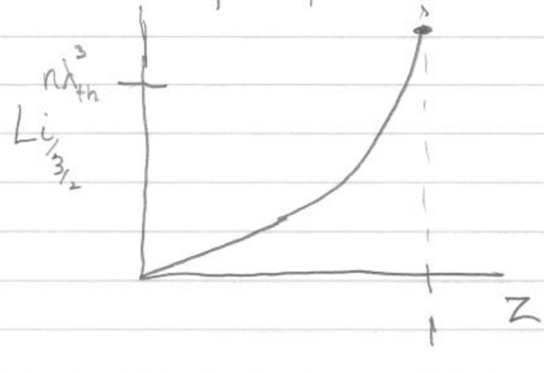
$$\boxed{N = V \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} Li_{3/2}\left(e^{\beta\mu}\right)}$$

$$\frac{1}{\lambda_{th}^3} = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}$$

$$U = \frac{V m^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^\infty dE \frac{E\sqrt{E}}{e^{\beta(E-\mu)} - 1} = \frac{V m^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \left(\frac{1}{\beta}\right)^{5/2} \Gamma\left(\tfrac{5}{2}\right) Li_{5/2}\left(e^{\beta\mu}\right)$$

$$\boxed{U = \frac{3}{2} N k T \frac{Li_{5/2}\left(e^{\beta\mu}\right)}{Li_{3/2}\left(e^{\beta\mu}\right)}}$$

(b) $T \to 0$, $\beta \to \infty$, $\mu < 0$, $e^{\beta\mu} \to ?$



at T_c, $e^{\beta\mu} \to 1$

$$\boxed{T_c = \frac{m}{2\pi\hbar^2 k_B}\left(\frac{Li_{3/2}(1)\, V}{N}\right)^{2/3}}$$

$$N = V\left(\frac{m}{2\pi\hbar^2\beta_c}\right)^{3/2} Li_{3/2}(1), \quad \beta_c = \frac{1}{k_B T_c}$$

Below T_c we must add gd. state 'condensate' separately as a distinct $E = 0$ state.

$$\left.\begin{aligned} &N = N_{above} + f(E=0) \\ &\underline{\underline{T<T_c}} \quad \boxed{N_1 = \frac{V}{\lambda_{th}^3} Li_{3/2}(\underbrace{e^{\beta\mu}}_{\sim 1}) + \frac{e^{\beta\mu}}{1 - e^{\beta\mu}}} \end{aligned}\right\} T < T_c$$

~~$T<T_c$~~

$$\boxed{U = \frac{3}{2} NkT \frac{Li_{5/2}(e^{\beta\mu})}{Li_{3/2}(e^{\beta\mu})} \xrightarrow[T<T_c]{} \frac{3}{2}\left[\frac{Li_{5/2}(1)}{Li_{3/2}(1)}\right] N_1 kT}$$

$$N \xrightarrow[T<T_c]{} \sim \underbrace{\frac{V}{\lambda_{th}^3} Li_{3/2}(1)}_{N_1 \propto T^{3/2}} - \frac{1}{\beta\mu} \quad \text{or} \quad \mu \simeq \frac{-1}{[N - N_1]\beta}$$