Microstate multiplicity

$$\Omega(N,n) = \frac{N!}{n!(N-n)!} = \binom{N}{n}$$

- # of ways to get n "heads" tossing N "coins"

Stirling's approximation: $N! \sim \sqrt{2\pi N}\, N^N e^{-N}$

$$\ln(\Omega) = \ln(N!) - \ln(n!) - \ln((N-n)!)$$

$$\sim N\ln N - \cancel{N} + \mathcal{O}(\ln N)$$

$$- n\ln n + \cancel{n} - (N-n)\ln(N-n) + \cancel{N-n} + \cdots$$

$$\xrightarrow[n=\frac{N}{2}+y]{} N\ln N - \left(\tfrac{N}{2}+y\right)\ln\left(\tfrac{N}{2}+y\right) - \left(\tfrac{N}{2}-y\right)\ln\left(\tfrac{N}{2}-y\right) + \cdots$$

$$= \frac{N}{2}\ln N^2 - \frac{N}{2}\ln\left(\frac{N^2}{4}-y^2\right) + y\ln\left(\frac{N/2-y}{N/2+y}\right) + \cdots$$

$$= -\frac{N}{2}\ln\left(\frac{1}{4}-\frac{y^2}{N^2}\right) + y\ln\left(1-\frac{2y}{N}\right) - y\ln\left(1+\frac{2y}{N}\right)$$

$$\xrightarrow[\ln(1+x) = x + \mathcal{O}(x^2)]{} -\frac{N}{2}\ln\left(\frac{1}{4}\right) - \frac{N}{2}\left(-\frac{4y^2}{N^2}\right) - \frac{2y^2}{N} - \frac{2y^2}{N}$$

$$= -\frac{N}{2}\ln\left(\frac{1}{4}\right) - \frac{2y^2}{N}$$

$$\Omega\left(N, \tfrac{N}{2}+y\right) \sim \left(\frac{1}{4}\right)^{-N/2} e^{-2y^2/N}$$

$$\sim 2^N e^{-2y^2/N}$$