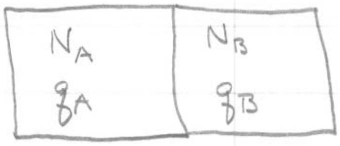
Two Einstein Solids in "Contact"



$$N = N_A + N_B \qquad q = q_A + q_B$$

$$q_A \gg N_A \gg 1 \;, \quad q_B \gg N_B \gg 1$$

What is the most probable distribution of energy (q) between solids (q_A, q_B).

$$\Omega = \Omega_A \times \Omega_B \sim \left(\frac{e q_A}{N_A}\right)^{N_A} \left(\frac{e q_B}{N_B}\right)^{N_B}$$

} see book for $q \gg N$ approx.

To find max., maximize $\ln \Omega$, subject to constraint $\rightarrow$ let $q_B = q - q_A$.

$$\ln \Omega \sim N_A \ln\left(\frac{e}{N_A}\right) + N_A \ln(q) + N_B \ln\left(\frac{e}{N_B}\right) + N_B \ln(q - q_A)$$

$$\frac{d}{dq_A} \ln \Omega \sim \frac{N_A}{q_A} - \frac{N_B}{q - q_A} \xrightarrow[\text{max}]{} 0$$

$$\hookrightarrow \quad q_A = \frac{N_A}{N} q \;, \quad q_B = \frac{N_B}{N} q$$

- This is the most likely division, but it is also very unlikely.

Consider distributions around the most likely:

$$q_A = \frac{N_A}{N} q + x \quad , \quad q_B = \frac{N_B}{N} q - x$$

Note these sum to q.

$$N_A \ln q_A = N_A \ln\left(\frac{N_A}{N} q + x\right) = N_A \ln\left(\frac{N_A}{N} q\right) + N_A \ln\left(1 + \frac{Nx}{N_A q}\right)$$

$$= N_A \ln\left(\frac{N_A}{N} q\right) + N_A \left[\frac{Nx}{N_A q} - \frac{1}{2}\left(\frac{Nx}{N_A q}\right)^2 + \cdots\right]$$

$$\sim N_A \ln\left(\frac{N_A}{N} q\right) + \frac{Nx}{q} - \frac{N^2 x^2}{2 N_A q^2}$$

use
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$$N_B \ln q_B \sim N_B \ln\left(\frac{N_B}{N} q\right) - \frac{Nx}{q} - \frac{N^2 x^2}{2 N_B q^2}$$

$$\Rightarrow \ln \Omega \sim N \ln\left(\frac{eq}{N}\right) - \frac{N^3}{2 N_A N_B} \frac{x^2}{q^2}$$

Here I used $\frac{1}{N_A} + \frac{1}{N_B} = \frac{N}{N_A N_B}$

$$\Omega \sim \left(\frac{eq}{N}\right)^N \exp\left[-\frac{N^3}{2 N_A N_B q^2} x^2\right]$$

This can be rewritten:

$$\Omega \sim \left(\frac{eq}{N}\right)^N \exp\left[-\frac{N N_A N_B}{2 q^2}\left(\frac{q_A}{N_A} - \frac{q_B}{N_B}\right)^2\right]$$