PHYSICS 621 **Autumn 2007** **Exam 2**

Name: Solutions

Do NOT panic if you can not finish both problems. Do as much as you can.

Show your work. Results without any derivation do not receive credit. **Derive all results starting from the equations below.** If you have memorized an equation that results from these four, derive it for full credit. Use this page for scratch work.

$\Delta U = Q + W \qquad U_{\text{thermal}} = \frac{1}{2}NfkT \qquad PV = NkT \qquad C = \frac{Q}{\Delta T}$

$S = k\ \ln(\Omega) \qquad \frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V}$

$\ln(n!) \approx n\ \ln(n) - n \qquad \ln(1+x) = x - \frac{x^2}{2} + \cdots$

two state system: $\Omega = \frac{N!}{n_1!n_2!}$, where $N = n_1 + n_2$

Einstein solid: $\Omega = \frac{(q+N-1)!}{q!(N-1)!}$

ideal gas: $\Omega = f(N)V^N U^{3N/2}$

1) (40 pts) An ideal atomic gas A contains N atoms, occupies a volume V and is at an initial temperature T_0. It is placed in contact with an identical atomic gas B that contains $2N$ atoms in a volume V (same as gas A) at temperature $2T_0$. (a) What is the final temperature, T_f, of the system when A and B equilibrate? (b) What are the changes in the entropies of the two gases, ΔS_A and ΔS_B? Does the total entropy increase? Give your answers in terms of N, V and T_0 or variables that you clearly define in terms of these.

A	B
N, V, T_0	$2N, V, 2T_0$

$$U_{A0} = \frac{3}{2} N k T_0 \qquad U_{B0} = 6 N k T_0$$

$$S_A = k \ln\left[F(N_A)\, U_A^{3N_A/2}\, V^{N_A/2}\right], \quad N_A = N$$

$$S_B = k \ln\left[F(N_B)\, U_B^{3N_B/2}\, V^{N_B/2}\right], \quad N_B = 2N$$

a) at equilibrium: $\dfrac{\partial S_A}{\partial U_A} = \dfrac{\partial S_B}{\partial U_B} \Rightarrow \dfrac{3N_A k}{2}\dfrac{1}{U_{Af}} = \dfrac{3N_B k}{2}\dfrac{1}{U_{Bf}}$

$\dfrac{U_{Af}}{N_A} = \dfrac{U_{Bf}}{N_B}$ or $U_{Af} = \dfrac{1}{2} U_{Bf}$ and $U_{Af} + U_{Bf} = U_{A0} + U_{B0}$

$$3U_{Af} = U_{A0} + U_{B0} \Rightarrow 3\left(\tfrac{3}{2} N k T_f\right) = \tfrac{15}{2} N k T_0 \qquad \boxed{T_f = \tfrac{5}{3} T_0}$$

b) $$\Delta S_A = \frac{3N_A k}{2} \ln\left(\frac{U_{Af}}{U_{A0}}\right) = \frac{3Nk}{2} \ln\left(\frac{T_f}{T_0}\right) = \frac{3Nk}{2} \ln\left(\frac{5}{3}\right)$$

$$\Delta S_B = \frac{3N_B k}{2} \ln\left(\frac{U_{Bf}}{U_{B0}}\right) = \frac{6Nk}{2} \ln\left(\frac{T_f}{2T_0}\right) = \frac{6Nk}{2} \ln\left(\frac{5}{6}\right)$$

$$\Delta S = \frac{3Nk}{2}\left[\ln\left(\frac{5}{3}\right) + 2\ln\left(\frac{5}{6}\right)\right] = \frac{3Nk}{2} \ln\left(\frac{5}{3} \times \frac{25}{36}\right)$$

$$= \frac{3Nk}{2} \ln\left(\frac{125}{108}\right) > 0 \checkmark$$

2) (60 pts) Two Einstein solids, A and B, with $N_A = e^{23}$ and $N_B = e^{21}$, $q_{A0} = e^{10}$ and $q_{B0} = e^{11}$, interact and come to equilibrium. Assume $U = q\epsilon$. (a) What are the most likely values for q_{Af} and q_{Bf} after they reach equilibrium? (b) Compute the temperatures T_{A0}, T_{B0} and T_f. Use approximations such as Stirling's approximation wherever final results will be precise to many digits. The subscripts refer to initial and final values.

A	B
N_A, q_A	N_B, q_B

$$U_{A0} = q_{A0}\epsilon \ , \ U_{B0} = q_{B0}\epsilon$$

$$S_A = k \ln\left[\frac{(q_A+N_A-1)!}{q_A!\,(N_A-1)!}\right] \simeq k\left\{(q_A+N_A)\ln(q_A+N_A) - q_A\ln(q_A) - N_A\ln(N_A)\right\}$$

$$\frac{\partial S_A}{\partial q_A} \simeq k\left\{\ln(q_A+N_A) + \cancel{1} - \ln(q_A) - \cancel{1}\right\} = k\ln\left(1+\frac{N_A}{q_A}\right) \simeq k\ln\left(\frac{N_A}{q_A}\right)$$

a) $\dfrac{\partial S_A}{\partial q_A} = \dfrac{\partial S_B}{\partial q_B}$ at equilibrium $\Rightarrow \dfrac{q_{AF}}{N_A} = \dfrac{q_{BF}}{N_B}$

and $q_{AF} + q_{BF} = q_{A0} + q_{B0}$, $\dfrac{q_{AF}}{N_A} = \dfrac{q_{A0}+q_{B0}-q_{AF}}{N_B}$

$$\frac{N_A+N_B}{N_A N_B}\, q_{AF} = \frac{q_{A0}+q_{B0}}{N_B}$$

$$q_{AF} = \frac{N_A}{N_A+N_B}\left(q_{A0}+q_{B0}\right) = \frac{e^{23}}{e^{23}+e^{21}}\left(e^{10}+e^{11}\right)$$

$$\boxed{q_{AF} = \frac{e^{12}(1+e)}{1+e^2}} \qquad q_{BF} = \frac{N_B}{N_A+N_B}\left(q_{A0}+q_{B0}\right) \qquad \boxed{q_{BF} = \frac{e^{10}(1+e)}{1+e^2}}$$

b) $$\frac{1}{T_A} = \frac{\partial S_A}{\partial U_A} = \frac{1}{\epsilon}\frac{\partial S_A}{\partial q_A} \simeq \frac{k}{\epsilon}\ln\left(\frac{N_A}{q_A}\right)$$

$$\frac{1}{T_{A0}} \simeq \frac{k}{\epsilon}\ln(e^{13}) \qquad \boxed{T_{A0} \simeq \frac{\epsilon}{13k}} \qquad \frac{1}{T_B} \simeq \frac{k}{\epsilon}\ln(e^{10}) \qquad \boxed{T_{B0} \simeq \frac{\epsilon}{10k}}$$

$$\frac{1}{T_F} \simeq \frac{k}{\epsilon}\ln\left(\frac{e^{23}(1+e^2)}{e^{12}(1+e)}\right) \qquad \boxed{T_f \simeq \frac{\epsilon}{k\left[11+\ln\left(\frac{1+e^2}{1+e}\right)\right]}}$$