Physics 594 } AU09 Exam #1 Solutions

1) $\hat{H} = \hbar\omega\sigma_z \Rightarrow \underline{\underline{\hat{H}|\uparrow\rangle = \hbar\omega|\uparrow\rangle, \; \hat{H}|\downarrow\rangle = -\hbar\omega|\downarrow\rangle}}$

- measure S_y at $t=0$ & get $\hbar/2$, implies

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \quad \text{or} \quad \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)$$

The easiest way to find $|\psi(t)\rangle$:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\longrightarrow \quad |\psi(t)\rangle = \exp\left\{-\frac{i\hat{H}t}{\hbar}\right\}|\psi(0)\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\hat{H}t/\hbar}|\uparrow\rangle + i e^{-i\hat{H}t/\hbar}|\downarrow\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|\uparrow\rangle + i e^{i\omega t}|\downarrow\rangle\right]$$

Alternative: $|\psi(t)\rangle = \begin{pmatrix}c_1(t)\\ c_2(t)\end{pmatrix}$ } $\boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega t}\\ ie^{i\omega t}\end{pmatrix}}$

$$i\hbar\frac{d}{dt}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \hbar\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \hbar\omega\begin{pmatrix}c_1\\ -c_2\end{pmatrix}$$

$$i\hbar\frac{d}{dt}c_1 = \hbar\omega c_1 \Rightarrow c_1(t) = c_1(0)e^{-i\omega t} = \frac{1}{\sqrt{2}}e^{-i\omega t}$$

$$i\hbar\frac{d}{dt}c_2 = -\hbar\omega c_2 \Rightarrow c_2(t) = c_2(0)e^{i\omega t} = \frac{i}{\sqrt{2}}e^{i\omega t}$$

$t = t_1$, measure S_x. The possible results are $\pm \hbar/2$.

$$P(+\hbar/2) = \left| \langle \uparrow_x | \psi(t_1) \rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1 \;\; 1) \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega t_1} \\ i e^{i\omega t_1} \end{pmatrix} \right|^2 = \frac{1}{4} \left| e^{-i\omega t_1} + i e^{i\omega t_1} \right|^2$$

$$= \frac{1}{4} \left(e^{-i\omega t_1} + i e^{i\omega t_1} \right) \left(e^{i\omega t_1} - i e^{-i\omega t_1} \right)$$

$$= \frac{1}{4} \left(2 + i e^{2i\omega t_1} - i e^{-2i\omega t_1} \right)$$

$e^{i\phi} - e^{-i\phi} = 2i \sin\phi$

$$= \frac{1}{4} \left[2 + i \left(2i \sin(2\omega t_1) \right) \right]$$

$$= \frac{1}{2} - \frac{1}{2} \sin(2\omega t_1)$$

$$P(-\hbar/2) = \left| \langle \downarrow_x | \psi(t) \rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1 \;\; -1) \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega t_1} \\ i e^{i\omega t_1} \end{pmatrix} \right|^2 = \frac{1}{4} \left| e^{-i\omega t_1} - i e^{i\omega t_1} \right|^2$$

$$= \frac{1}{4} \left(e^{-i\omega t_1} - i e^{i\omega t_1} \right) \left(e^{i\omega t_1} + i e^{-i\omega t_1} \right)$$

$$= \frac{1}{4} \left(2 - i e^{2i\omega t_1} + i e^{-2i\omega t_1} \right)$$

$$= \frac{1}{4} \left[2 - i \left(2i \sin(2\omega t_1) \right) \right]$$

$$= \frac{1}{2} + \frac{1}{2} \sin(2\omega t_1)$$

Probabilities sum to 1. ✓

2) initial: 

$\lambda\nu = c$: $\quad p_\gamma = \frac{h}{\lambda}, \; E_\gamma = h\nu = \frac{hc}{\lambda} \qquad p_M = 0, \; E_M = 0$

↑ nonrelativistic

Final: 

$p_\gamma' = -\frac{h}{2\lambda}, \; E_\gamma' = \frac{hc}{2\lambda} \qquad p_M', \; E_M' = \frac{p_M'^2}{2M}$

↑ direction

momentum conservation: $\frac{h}{\lambda} + 0 = -\frac{h}{2\lambda} + p_M' \quad (i)$

energy conservation: $\frac{hc}{\lambda} + 0 = \frac{hc}{2\lambda} + \frac{p_M'^2}{2M} \quad (ii)$

(i) $p_M' = \frac{3h}{2\lambda}$

ii) $\underbrace{\frac{p_M'^2}{2M} = \frac{1}{2M}\left(\frac{3h}{2\lambda}\right)^2 = \frac{hc}{2\lambda}}$

$M = \frac{9h^2}{8\lambda^2} \cdot \frac{2\lambda}{hc} = \frac{9h}{4\lambda c}$

Note that this is actually wrong.

SEE NEXT PAGE

I did not give you the equation from the book, $\lambda' - \lambda = \frac{h}{Mc}(1-\cos\theta)$.

I graded this test liberally but will give no credit or little credit in the future for jumping to a result derived in the book from memory.

Extra credit: If you use this eq., with $\theta = \pi$, you get:

$$2\lambda - \lambda = \lambda = \frac{h}{Mc}(2)$$

$$M = \frac{2h}{\lambda c}$$

Why is this completely different from $\frac{9h}{4\lambda c}$?