Physics 594 Quiz #1 Solution

Spin-½ particle is placed in a magnetic field, $\vec{B} = B\hat{z}$, so that the "Hamiltonian" (from which we obtain energies in QM) is:

$$\hat{H} = \mu_B B \sigma_z \, , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

At time $t=0$, the x-component of spin, S_x, is measured and found to be $\hbar/2$. At a later time $t = t_1$, S_y is measured. What are the possible results and their corresponding probabilities?

<u>t = 0</u>: $S_x |\uparrow_x\rangle = \frac{\hbar}{2} |\uparrow_x\rangle \, , \quad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow \quad |\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>time evolution</u>: Given $|\psi(0)\rangle = \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}$ I showed

$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$ implies:

$$|\psi(t)\rangle = \begin{pmatrix} c_1(0) e^{-i\omega t} \\ c_2(0) e^{i\omega t} \end{pmatrix} \, , \quad \omega \equiv \frac{\mu_B B}{\hbar}$$

In this problem $c_1(0) = c_2(0) = \frac{1}{\sqrt{2}}$.

<u>measurent at $t = t_1$</u>: We need the eigenvalues and eigenstates of $S_y = \hbar/2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. As stated in class, the eigenvalues are $\pm \hbar/2$, so these are the possible outcomes. To find the eigenstates:

$$S_y |\uparrow_y\rangle = \hbar/2 |\uparrow_y\rangle \Rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \hbar/2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\Rightarrow -ib_2 = b_1 \text{ and } ib_1 = b_2$$ (same because eigenvalue is there)

$$\Rightarrow |\uparrow_y\rangle = \begin{pmatrix} b_1 \\ ib_1 \end{pmatrix}$$

Normalize: $|b_1|^2 + |b_1|^2 = 1$, let $b_1 = \frac{1}{\sqrt{2}}$.

$$|\uparrow_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

For $-\hbar/2$ we get $-ib_2 = -b_1$, leading to:

$$|\downarrow_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

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From the postulates $\mathcal{P}(a_n) = |\langle a_n | \psi(t)\rangle|^2$:

$$\mathcal{P}\left(+\tfrac{\hbar}{2}\right) = \left| \langle \uparrow_y | \psi(t) \rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega t} \\ \frac{1}{\sqrt{2}} e^{i\omega t} \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{2} \left(e^{-i\omega t} - i e^{i\omega t} \right) \right|^2$$

$$= \frac{1}{4} \left(e^{-i\omega t} - i e^{i\omega t} \right)\left(e^{i\omega t} + i e^{-i\omega t} \right)$$

$$= \frac{1}{4} \left(1 + 1 + i e^{-2i\omega t} - i e^{2i\omega t} \right)$$

$$= \frac{1}{4} \left[2 + i \left(e^{-2i\omega t} - e^{2i\omega t} \right) \right]$$

$$\xrightarrow[e^{-i\phi} - e^{i\phi} = -2i \sin\phi]{} \quad \frac{1}{4} \left[2 + 2\sin(2\omega t) \right]$$

$$\mathcal{P}\left(+\frac{\hbar}{2}\right) = \frac{1}{2}\left(1 + \sin(2\omega t)\right)$$

$$\rightarrow \quad \mathcal{P}\left(-\tfrac{\hbar}{2}\right) = \frac{1}{2}\left(1 - \sin(2\omega t)\right)$$