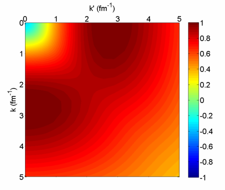
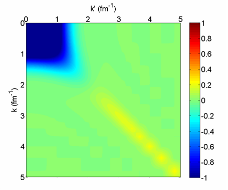
Factorization of Operators Evolved with the Similarity Renormalization Group



Eric R. Anderson

Department of Physics
The Ohio State University

February 16, 2010



T · H · E OHIO STATE UNIVERSITY

<u>In Collaboration with</u>: S.K Bogner, R.J. Furnstahl, E.D. Jurgenson, & R.J. Perry

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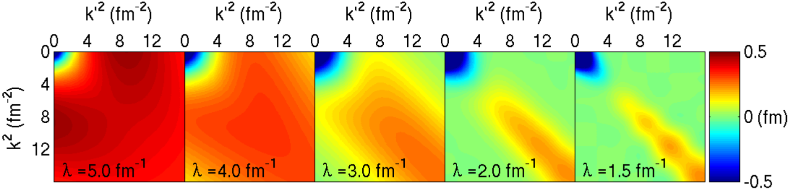
SRG: A Brief Review

- **The Similarity Renormalization Group (SRG)**
  $\rightarrow$ provides a means to systematically evolve computationally difficult **potentials and operators**

$$O_s = U_s O_{s=0} U_s^\dagger \quad \Longleftrightarrow \quad \frac{dO_s}{ds} = [\eta_s, O_s] = [[T_{rel}, H_s], O_s]$$

- Hamiltonian Operator $\rightarrow$ driven toward diagonal or decoupled form



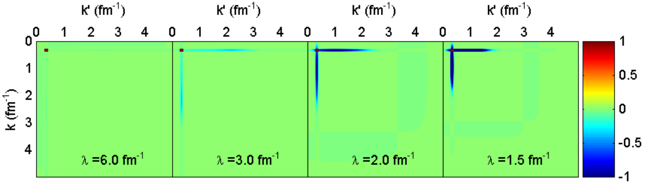
$^3S_1$ **AV18 Evolution**

- Can unitarily evolve operators consistent with any initial potential, in general
- Issues: – Decoupling
  – Factorization

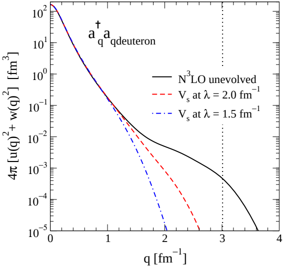
Note:
$\lambda = \frac{1}{s^{1/4}}\ \mathrm{fm}^{-1}$

High and Low Momentum operators in the Deuteron

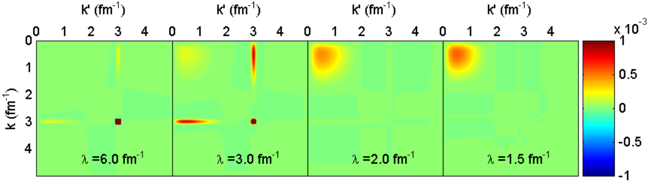
- **Integrand** of $\langle\psi_d|\, U^\dagger (U a_q^\dagger a_q U^\dagger) U \,|\psi_d\rangle$ for $q = 0.34\,\text{fm}^{-1}$



- Momentum Distribution



- **Integrand** for $q = 3.02\,\text{fm}^{-1}$



- **Decoupling** $\leftrightarrow$ High momentum components suppressed
- Integrated value does not change, but nature of operator does
- Similar for other operators: $\langle r^2\rangle$, $\langle Q_d\rangle$, & $\langle \frac{1}{r}\rangle$

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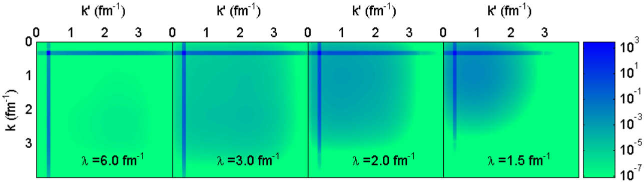

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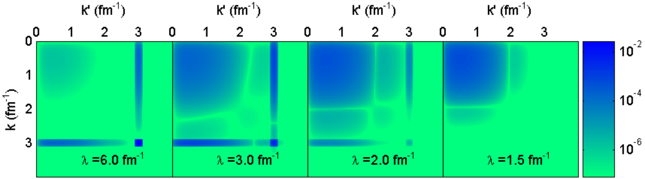

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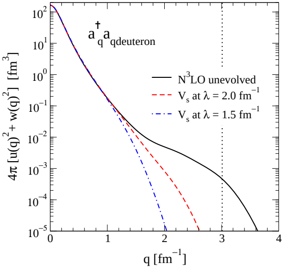

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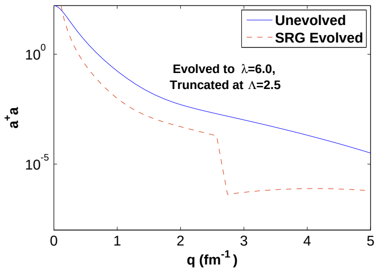
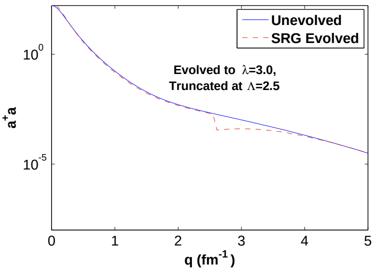
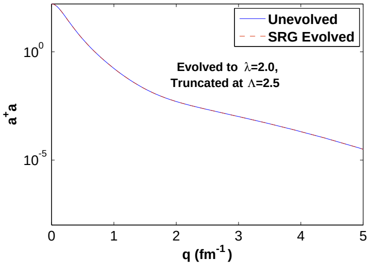
Demonstration of Decoupling In Expectation Values

- **Evolve** Hamiltonian & operators **to** λ **in full space** $\rightarrow$ **TRUNCATE at** Λ:





- Momentum distribution
- $\Lambda = 2.5\,\mathrm{fm}^{-1}$
  $\lambda = 6.0\,\mathrm{fm}^{-1}, 3.0\,\mathrm{fm}^{-1}$, and $2.0\,\mathrm{fm}^{-1}$
- Decoupling **for all q** is successful when $\lambda < \Lambda$
- Calculated with AV18 potential.



Factorization

Motivation: The **Operator Product Expansion** (OPE) of Nonrelativistic Wave Function (Lepage)

$$\Psi_{true}(r) = \overline{\gamma}(r) \int d\mathbf{r}' \, \Psi_{eff} \delta_a(\mathbf{r}')$$
$$+ \overline{\eta}(r) a^2 \int d\mathbf{r}' \, \Psi_{eff} \nabla^2 \delta_a(\mathbf{r}') + \mathcal{O}(a^4)$$

Similarly, in momentum space

$$\Psi^{\infty}_{\alpha}(q) \approx \gamma^{\lambda}(q) \int_0^{\lambda} p^2 dp \; Z(\lambda) \Psi^{\lambda}_{\alpha}(p)$$
$$+ \eta^{\lambda}(q) \int_0^{\lambda} p^2 dp \, p^2 \, Z(\lambda) \, \Psi^{\lambda}_{\alpha}(p)$$

$\rightarrow$ By projecting the nuclear potential in momentum subspace, recover OPE via:

$$\gamma^{\lambda}(q) \equiv -\int_{\lambda}^{\infty} q'^2 dq' \, \langle q | \frac{1}{Q_{\lambda} H^{\infty} Q_{\lambda}} | q' \rangle V^{\infty}(q', 0)$$

$$\eta^{\lambda}(q) \equiv -\int_{\lambda}^{\infty} q'^2 dq' \, \langle q | \frac{1}{Q_{\lambda} H^{\infty} Q_{\lambda}} | q' \rangle \frac{\partial^2}{\partial p^2} V^{\infty}(q', p)|_{p^2=0}$$

Questions:

- $\langle \psi_d | \, U_{\lambda} a^{\dagger}_q a_q U^{\dagger}_{\lambda} \, | \psi_d \rangle$ is independent of λ. What is the nature of $U_{\lambda} a^{\dagger}_q a_q U^{\dagger}_{\lambda}$?
- If $k < \lambda$ and $q \gg \lambda \Longrightarrow$ **factorization**?

$$U_{\lambda} \rightarrow K_{\lambda}(k) Q_{\lambda}(q)$$

Can construct transformation directly:

$$U_{\lambda}(k, q) = \sum_{\alpha} \langle k | \psi^{\lambda}_{\alpha} \rangle \langle \psi^{\infty}_{\alpha} | q \rangle$$

$$\rightarrow \left[\sum_{\alpha}^{\alpha_{low}} \langle k | \psi^{\lambda}_{\alpha} \rangle \int_0^{\lambda} p^2 dp \; Z(\lambda) \Psi^{\lambda}_{\alpha}(p) \right] \gamma^{\lambda}(q)$$

$$\Rightarrow U_{\lambda}(k, q) \approx \; K_{\lambda}(k) Q_{\lambda}(q)$$

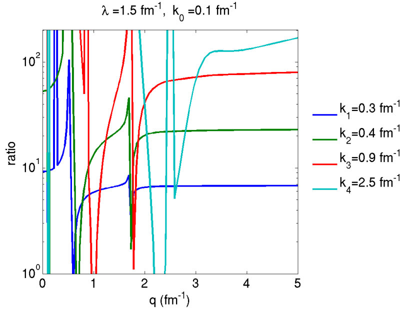
Numerical Factorization

- A **preliminary test of factorization** in U can be made by assuming

$$\frac{U_\lambda(k_i,q)}{U_\lambda(k_0,q)} \to \frac{K_\lambda(k_i)Q_\lambda(q)}{K_\lambda(k_0)Q_\lambda(q)},$$

we know that for $q \gg \lambda \Rightarrow \frac{K_\lambda(k_i)}{K_\lambda(k_0)}$.

- As shown below, one can infer this behavior from the plateaus for $q \gtrsim 2\text{fm}^{-1}$



- **Singular Value Decomposition (SVD)** → tool to quantitatively analyze the extent to which U factorizes
  → The SVD can be expressed as an outer product expansion

$$G = \sum_i^r d_i \vec{u}_i \vec{v}_i^t$$

where r is the rank and the d_i are the singular values (in order of decreasing value).

- **Evidence**: Shown below at $\lambda = 2\,\text{fm}^{-1}$, for $q > \lambda$ and $k < \lambda$

	$^1S_0$		
Potential	**d_1**	**d_2**	**d_3**
AV18	0.763	0.033	0.007
N3LO 500 MeV	1.423	0.221	0.015
N3LO 550/600 MeV	3.074	0.380	0.061
	$^3S_1$–$^3S_1$		
AV18	0.671	0.015	0.008
N3LO 500 MeV	1.873	0.225	0.044
N3LO 550/600 MeV	4.195	0.587	0.089

Practical Use

- From **Decoupling**: write

$$\langle\psi_\lambda|\, U_\lambda \widehat{O}\, U_\lambda^\dagger\, |\psi_\lambda\rangle \cong$$

$$\int_0^\lambda dk' \int_0^\infty dq' \int_0^\infty dq \int_0^\lambda dk\; \psi_\lambda^\dagger(k') U_\lambda(k',q') \widehat{O}(q',q) U_\lambda(q,k) \psi_\lambda(k)$$

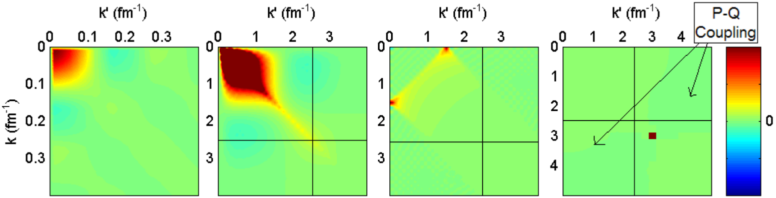
- Using **Factorization**: set $U_\lambda(k,q) \to K_\lambda(k) Q_\lambda(q)$, where $k < \lambda$ and $q \gg \lambda$.

$$\Longrightarrow \int_0^\lambda \int_0^\lambda \psi_\lambda^\dagger(k') \left[\int_0^\lambda \int_0^\lambda \underbrace{U_\lambda(k',q')\widehat{O}(q',q)U_\lambda(q,k)} + I_{QOQ}\, \underbrace{K_\lambda(k')K_\lambda(k)} \right] \psi_\lambda(k)$$

Low Momentum Structure

where $I_{QOQ} = \int_\lambda^\infty dq' \int_\lambda^\infty dq \left[Q_\lambda(q') \widehat{O}(q',q) Q_\lambda(q) \right]$ $\leftarrow$ *Universal*

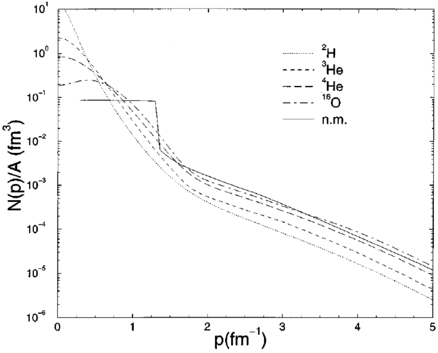
– *Valid* when initial operators *weakly couple high and low momentum*, e.g.,



r^2 $\quad$ $\frac{1}{r}$ $\quad$ $G_C(q = 3.02\,\text{fm}^{-1})$ $\quad$ $a_q^\dagger a_q(q = 3.02\,\text{fm}^{-1})$

Factorization in Few-Body Nuclei

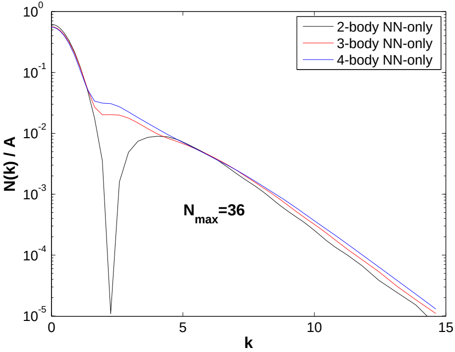
- **Variational Monte Carlo Calculation**
  → Using AV14 NN potential



From Pieper, Wiringa, and Pandharipande (1992).

- Possible explanation of scaling behavior
  → Results from dominance of NN potential and short-range correlations (Frankfurt, et al.)

- **1D few-body HO space calculation**
  → System of A bosons interacting via a model potential



→ A Test Bed for 3D NCSM calculations:

- Alternative explanation of scaling behavior
  → Results from *factorization*

$$\int_0^\lambda \int_0^\lambda \psi_\lambda^\dagger(k') \left[I_{QOQ} K_\lambda(k') K_\lambda(k) \right] \psi_\lambda(k)$$

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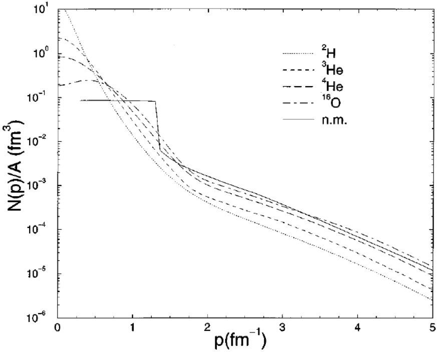

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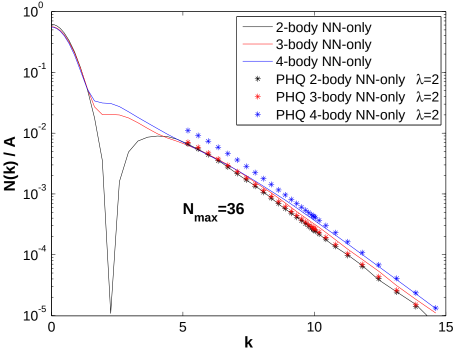

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Conclusion/Outlook

Summary:

- Nuclear operators can be consistently evolved and calculated with the SRG
- Factorization: $U_\lambda(k, q) \to K_\lambda(k) Q_\lambda(q)$
- Factorization and few-body model calculation

Outlook:

- Scaling of many-body operators
- 3D in Harmonic Oscillator basis
- Factorization of other operators
  –Electromagnetic interactions, etc.

The End