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# Exercises #4

This set concerns the physical pendulum, which satisfies the equation of motion

$$\frac{d^2}{dt^2}\theta(t) = -\omega^2 \sin[\theta(t)].$$

The pendulum is released at rest at its maximum amplitude  $\alpha$ , and executes a period motion. If the amplitude is small, we can approximate  $\sin(\theta) \approx \theta$ , in which case we have the simple harmonic oscillator with period  $T = 2\pi/\omega$ . More generally, dimensional analysis tells us that the period is a function

$$T(\theta) = \frac{2\pi}{\omega} f(\alpha)$$

where  $f(0) = 1$ . For the remainder we will set  $\omega = 1$ .

•**Problem 1.** Solve the differential equation for  $\alpha = 0.1$ , and plot the resulting  $\theta(t)$  for about a period. Repeat for a few  $\alpha$  in the range .1 to 3.0, plotting several curves  $\theta(t)/\alpha$  in one figure.

•**Problem 2.** Write a function to numerically determine the period as a function of the amplitude. Do this by solving for the position of the first zero in  $\theta(t)$ .

•**Problem 3.** Study the function  $f(\alpha)$  for small  $\alpha$  and estimate the coefficient  $c$  which gives the leading correction:

$$f(\alpha) = 1 + c\alpha^2 + \dots$$

•**Problem 4.** To get a more exact expression for the period it helps to have the following insight. There is an integral of the motion, i.e. the energy, which reduces the equation to first order. That is, since energy is conserved, we know not only the acceleration, but also the velocity as a function of  $\theta$ :

$$E = \frac{1}{2}[\theta'(t)]^2 + \cos[\theta(t)] = \cos[\alpha]$$

We can use this insight to write the period as

$$T = 2\sqrt{2} \int_0^\alpha \frac{1}{\sqrt{\cos(\theta) - \cos(\alpha)}} d\theta$$

Write a new function which gets the period by doing this integral numerically. (When you have had enough coaxing Mathematica to do the integral, help it along by making a change of variables  $\theta = \alpha x$ , so the integral runs from 0 to 1.)

•**Problem 5.** Now obtain an exact expression for the coefficient  $c$  by expanding the integrand in powers of  $\alpha$  before integrating.

•**Problem 6.** Compute several terms in the series. Can you guess the pattern in the coefficients? Either way, write a function to compute the period by series, and check the result for  $\alpha = 1$ . For those with extra time and great patience, see how many terms you need for  $\alpha = 2$  and  $\alpha = 3$ .