Review Sheet for H131 Final Exam

Note: You are also responsible for everything you were supposed to know for the midterm exam, so go over the midterm review sheet as well!

What you should know about . . .

Angular Momentum [C13-14,N12-13]

1. Definition: $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ is position relative to some specified origin O and $\vec{p} = m\vec{v}$ is momentum of object. Note: $\vec{L}$ depends crucially on choice of reference point O!
   - Units are kg·m^2/s
   - For extended objects $\vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i = \sum_{i=1}^{N} \vec{r}_i \times m\vec{v}_i$
   - Cross product $\vec{u} \times \vec{w}$ is a *vector* whose direction is fixed by the right hand rule (it is perpendicular to both $\vec{u}$ and $\vec{w}$!) and whose magnitude is $|\vec{u} \times \vec{w}| = uw \sin\theta = = uw_\perp = u_\perp w$. ($\theta$ is the angle between the directions of $\vec{u}$ and $\vec{w}$.)
   - The components of $\vec{u} \times \vec{w}$ are calculated from
     $(\vec{u} \times \vec{w})_x = u_y w_z - u_z w_y$,
     $(\vec{u} \times \vec{w})_y = u_z w_x - u_x w_z$,
     $(\vec{u} \times \vec{w})_z = u_x w_y - u_y w_x$.
   - $\vec{u} \times \vec{w} = -\vec{w} \times \vec{u}$; the cross product of two parallel vectors vanishes.
2. $\vec{L} = I\vec{\omega}$ for a symmetrical rotating object. An object is symmetrical around an axis of rotation if for every particle there is an identical particle directly across the rotational axis at the same distance from it.
3. For an object that is both translating and rotating, $\vec{L} = \vec{L}^{\text{cm}} + \vec{L}^{\text{rot}} = \vec{r}_{\text{cm}} \times M\vec{v}_{\text{cm}} + \vec{L}^{\text{rot}}$.
   If the rotation is around a symmetry axis, $\vec{L}^{\text{rot}} = I\vec{\omega}$.
4. The total angular momentum of an isolated system is conserved: $\vec{L}_{1i} + \vec{L}_{2i} + \ldots = \vec{L}_{1f} + \vec{L}_{2f} + \ldots$.
5. Conservation of angular momentum $\vec{L} = \vec{r} \times m\vec{v}$ implies that planetary orbits are planar ($\vec{r} \perp \vec{L}$ must lie in a fixed plane $\perp \vec{L}$).

What you should know about . . .

Forces and Newton's Laws [N1,N3-N6]

1. Any potential energy $V(r)$ which depends on the separation r between two interacting objects generates a force $F_r = -dV/dr$ along the line connecting them. The magnitude of this force is $F = |dV/dr|$.
2. Forces are vectors and are added as vectors.
3. Please review the list of forces in Section N1.5. Some remarks:
   - Static friction force acts opposite to direction of motion with magnitude $F_{SF} \leq F_{SF,\max} = \mu_s F_N$ where F_N is the normal force (usually proportional to the object's weight $F_g = mg$). [N6.3]
   - Kinetic friction force acts opposite to direction of motion with magnitude $F_{KF} = \mu_k F_N$ ($\mu_k < \mu_s$). [N6.3]
   - Drag force acts opposite to direction of motion with magnitude $F_D = \frac{1}{2} C\rho A v^2$ [N6.4]. For small objects moving slowly through thick liquids $F_D \propto v$.

4. Two interacting objects feel forces of equal magnitude but opposite direction (Newton's third law). Such pairs of forces, arising from the same interaction, but acting on different obejects, are called "third-law pairs".

5. The net force $\vec{F}_{\rm net} = \vec{F}_1 + \vec{F}_2 + \ldots$ acting on an object is responsible for its acceleration according to Newton's second law: $\vec{F}_{\rm net} = m\vec{a}$.

6. For an extended object, all *internal* forces come as "third-law pairs" and add up to zero. It's acceleration is only caused by *external* forces.

7. In "free-body diagrams" one attaches force vectors to a sketch of the object under consideration at point where and in the direction in which they attack. This makes "third-law pairs" of forces acting between coupled objects easy to spot.

8. "Free-particle diagrams" exploit the fact that for translations an extended object can be idealized by a point mass located in its center of mass; correspondingly, all force vectors attack on the center of mass. This makes "third-law pairs" harder to spot, but gives a better pictorial idea of where the net force points. This is important if you want to check whether you have included all relevant forces.

9. A "net force diagram" explicitly constructs the net force vector from the vector sum of the individual forces.

10. Netwon's first law states that in the absence of external forces a body moves with constant velocity (i.e. *speed* and *direction* of its motion remain unchanged).

11. "Second-law pairs" of forces are force pairs acting on a single object, but arising from different types of interactions, which add up to zero because the motion of the object is constrained to have zero acceleration in certain directions. They arise routinely in linearly constrained motion.

What you should know about ...

Position, Velocity, and Acceleration [N1-N8,N10]

1. From the position $\vec{r}(t)$ of an object as a function of time ("trajectory") one obtains its velocity $\vec{v}(t)$ by differentiation with respect time: $\vec{v}(t) = d\vec{r}/dt$. A second time derivative gives its acceleration: $\vec{a}(t) = d\vec{v}/dt = d^2\vec{r}/dt^2$.

2. An object's velocity $v(t)$ is obtained from its acceleration $\vec{a}(t)$ by integration over time: $\vec{v}(t) - \vec{v}(t_0) = \int_{t_0}^{t} \vec{a}(t')dt'$.
   This implies a separate integration for each vector component. The object's position is obtained after a second time integration:
   $\vec{r}(t) - \vec{r}(t_0) = \int_{t_0}^{t} \vec{v}(t')dt'$.
   In most situations one can simply set $t_0 = 0$ without loss of generality.

3. Review your math skills on differentiation and integration [N2-4].
   (Powers, exponential functions, trigonometric functions, product rule, chain rule, ...)

4. In many practical situations the acceleration $\vec{a}(t)$ is determined from the forces acting on the object via Newton's second law: $m\,\vec{a}(t) = \vec{F}_{\rm net}(t)$.

5. For constant acceleration $\vec{a}(t) = \vec{a}_0 =$const. (i.e. constant net external force), the integrations in 2. above give the general solution for "simple projectile motion":

   - $\vec{v}(t) = \vec{v}_0 + \vec{a}_0\,t, \qquad \vec{r}(t) = \vec{r}_0 + \vec{v}_0\,t + \frac{1}{2}\vec{a}_0\,t^2.$

- In this case it is convenient to orient the reference frame in such a way that $\vec{a}_0$ points along the z axis. This means that the acceleration enters only in the z components, while the x, y components describe uniform motion with constant velocity (v_{x0}, v_{y0}).
- This means that we can further rotate the x and y axes such that $v_{y0} = 0$, and by shifting the origin we can also arrange that $y_0 = 0$. The solution then becomes
  $\vec{v}(t) = (v_{x0},\, 0,\, v_{z0} + a_0\, t)$
  $\vec{r}(t) = (x_0 + v_{x0}t,\, 0,\, z_0 + v_{z0}t + \frac{1}{2}a_0t^2)$
- If the constant acceleration is caused by gravity, $m\vec{a}_0 = \vec{F}_g = m\vec{g} = (0, 0, -mg)$, only the z-components of the equations for $\vec{r}(t)$ and $\vec{v}(t)$ contain interesting information. (In this case we must set $a_0 = -g$ in the above equations.) *All questions about how long or how far a simple projectile flies must and can be answered by first solving the motion in z-direction.*

6. Projectile motion with drag: terminal speed $v_T = \sqrt{(2mg)/(C\rho A)}$ [N10]. Velocity of freely falling projectile with drag: $v_z(t)/v_T = -\tanh(gt/v_T)$. Time needed to reach terminal speed: $t_T \approx 2v_T/g$.

What you should know about ...

Coupled Objects [N7]

1. The abundance of forces affecting coupled objects makes it necessary to use a complete notation of the type $\vec{F}_N^{A(B)}$ [N7].
2. An *ideal string* coupling two objects is a string which is completely massless, inextensible, and flexible. The tension forces exerted by each end of the string have equal magnitude and opposite direction. Two objects connected by an ideal string move exactly together.
3. An *ideal pulley* is massless and frictionless. It changes the direction of a string without affecting the magnitudes of the tension forces it exerts. It is advantageous to orient the reference frames for objects connected via a string which runs over a pulley differently, so that the acceleration for both objects points along the same coordinate axis.

What you should know about ...

Circularly Constrained Motion [N8,N12]

1. $\widehat{r}$, $\widehat{\phi}$ are unit vectors ("directionals") in the radial and tangential directions. The unit vector $\widehat{v}$ points parallel to the particle's velocity.
2. For objects moving along a circle with radius R:
   $\vec{r} = R\,\widehat{r}$ with constant R; $\vec{v} = v_\perp \widehat{v}$ with $v_\perp = R\,(d\phi/dt)$.
3. For uniform circular motion $v = |v_\perp| = R\omega$ is constant. The orbital period is then $T = (2\pi R)/v$.
4. The acceleration of an object moving around a circle is $\vec{a} = (dv/dt)\,\widehat{v} - (v^2/R)\,\widehat{r}$.
   The radial acceleration $a_r = v^2/R$ points towards the circle's center.
   For uniform circular motion the tangential acceleration $a_\phi = dv/dt$ is zero.
5. The radial acceleration required for circular motion can be provided entirely by the normal force when banking at the ideal banking angle $\tan\theta = v^2/(Rg)$ [N8].

What you should know about ...

Non-Inertial Reference Frames [N9]

1. Galilean transformation: If S, S' are two reference systems with axes oriented in the same directions, and $\vec{R}(t)$, $\vec{\beta}(t)$, $\vec{A}(t)$ is the displacement, velocity, and acceleration of S' relative to S, then the position, velocity, and acceleration of a given object measured in S and S' are related by
$\vec{r}(t) = \vec{r}\,'(t) + \vec{R}(t)$; $\quad \vec{v}\,'(t) = \vec{v}(t) - \vec{\beta}(t)$; $\quad \vec{a}\,'(t) = \vec{a}(t) - \vec{A}(t)$.

2. Inertial frames are reference frames in which an isolated object moves at constant velocity. Any frame which moves with constant velocity $\vec{\beta}$ relative to an inertial frame is again an inertial frame.

3. Non-inertial frames are frames S' which are accelerated relative to an inertial frame S: $\vec{A} \neq 0$.

4. Fictitious forces are forces which do not arise from an interaction but from the "inappropriate" application of Newton's second law in a non-inertial frame, thereby interpreting the acceleration $\vec{A}$ of the frame via $-m\vec{A} = \vec{F}_{\text{fictitious}}$ as a force. Examples of fictitious forces: "centrifugal force" when going around a curve, "inertial force" when braking.

5. In a freely falling system gravity seems to vanish since everything falls with the same acceleration along with the frame. We can ignore external gravitational forces if we use a freely falling frame.

6. The apparent force $-m\vec{A}$ in a constantly accelerating frame acts like gravity of effective strength $g_{\text{eff}} = A$ (so if there is gravity as well, $\vec{g}_{\text{eff}} = \vec{g} - \vec{A}$).

What you should know about ...

Oscillatory Motion [N11]

1. Ideal spring: potential energy $V_{sp}(x) = \frac{1}{2}k_s x^2$ (where $x = r - r_0$ is the extension of the spring away from its relaxed length r_0); spring force $F_{sp,x} = -dV_{sp}/dx = -k_s x$.

2. Newton's second law with spring force = harmonic oscillator equation: $d^2x/dt^2 = -\omega^2 x$ with $\omega = \sqrt{k_s/m}$.

3. General solution of harmonic oscillator equation: $x(t) = A\cos(\omega t + \theta)$. Amplitude $A > 0$ is the maximum extension $x_{\max}$ of spring. ω = rate of phase change; period of oscillation $T = 2\pi/\omega$, frequency $f = \omega/(2\pi)$. Phase constant θ determines where object is in its cycle at $t = 0$.

4. Velocity of oscillating object $v_x(t) = -A\omega\sin(\omega t + \theta)$. $A\omega = v_{\max}$ determines maximum speed of object.

5. A and θ fixed by initial conditions: $\theta = \tan^{-1}[-v_{x0}/(\omega x_0)]$; $\quad A = x_0/\cos\theta$.

6. For an object hanging from a spring, Newton's second law turns into the harmonic oscillator equation after shifting x by the static extension $x_{\text{eq}} = -mg/k_s$ of the spring due to the object's weight:
$x' = x - x_{\text{eq}} = x + mg/k_s$.

7. *Any* object or system near stable equilibrium performs harmonic oscillations when slightly displaced from its equilibrium state. Any restoring force takes the approximate form $F_x = -kx$ sufficiently close to an equilibrium state (i.e. for sufficiently small x). Example: simple pendulum (see next item).

8. Simple pendulum: object suspended from a fixed point by an ideal string.
Newton's equation for the simple pendulum: $d^2\phi/dt^2 = -(g/L)\sin\phi$. ($L$ = length of string, ϕ = angle of string relative to the vertical.) For $\phi \ll 1$ (small deviations from static equilibrium) $\sin\phi \approx \phi$, and the pendulum equation becomes a harmonic oscillator equation. In this low-angle limit, the period of the simple pendulum is $T = 2\pi\sqrt{L/g}$.

What you should know about . . .

Planetary Motion [N12,N13]

1. For two objects with masses m, M orbiting around each other under the influence of gravity, we have $\vec{R} = -(m/M)\vec{r}$, $\vec{V} = -(m/M)\vec{v}$, $K_M = (m/M)K_m$, $\vec{L}_M = (m/M)\vec{L}_m$. Here big (small) letters refer to the heavy (light) object, and all equations are correct in the frame where their center of mass sits at rest. These allow to determine everything for the heavy object once the dynamics of the light object has been solved.

2. If the "primary" M is much heavier than the "satellite", we can approximate $\vec{R} = \vec{V} = K_M = \vec{L}_M = 0$.

3. Kepler's laws: 1. planets orbit in ellipses, with sun at one focus; 2. a line from the sun to the planet sweeps out equal areas in equal times; 3. $T^2 \propto a^3$ where T=period and a=half of ellipse's largest width (semimajor axis)

4. Kepler's second law follows from angular momentum conservation: $dA/dt = \frac{1}{2}r^2\, d\theta/dt = L/(2m) =$ constant since $L = mr^2\, d\theta/dt$ is constant.

5. For circular orbits the orbital speed is constant, $v_{\text{circ}} = \sqrt{GM/R}$, and the period T and orbital radius R are related by $T^2 = (4\pi^2/GM)R^3$.

6. For elliptical orbits, this last relation changes simply to $T^2 = (4\pi^2/GM)a^3$.

7. Conic sections:

   - A conic section is defined by $r(\theta) = R/(1 + \epsilon\cos\theta)$
   - θ is the angle between the position vector from the *focus* and the line from the focus to point r_c of closest approach on the curve.
   - R is a constant length which specifies the scale of the curve; for planetary orbits it is related to the angular momentum around the focus of the ellipse by $R = (L/m)^2/(GM)$.
   - $\epsilon \geq 0$ is unitless and called the *eccentricity*.
     $\epsilon = 0$ for a circle;
     $\epsilon < 1$ for an ellipse;
     $\epsilon = 1$ for a parabola;
     $\epsilon > 1$ for a hyperbola.
   - For planetary orbits the eccentricity is related to the total energy $E = K + V$ (with $r = \infty$ as reference distance for V) and the angular momentum L around the focus by
     $\epsilon = \sqrt{1 + (2E/m)(L/m)^2/(GM)^2}$.
   - This implies:
     1. if $E < 0$ then $\epsilon < 1$ and the orbit is elliptical (bound);
     2. if $E = 0$ then $\epsilon = 1$ and the orbit is parabolic (just unbound);
     3. if $E > 0$ then $\epsilon > 1$ and the orbit is hyperbolic (unbound).
   - On ellipses, the closest and farthest distance from the focus are $r_c = R/(1+\epsilon)$, $r_f = R/(1-\epsilon)$. The semimajor axis $a = (r_c + r_f)/2 = R/(1-\epsilon^2)$, the semiminor axis $b = a\sqrt{1-\epsilon^2} = R/\sqrt{1-\epsilon^2}$, and the area $A = \pi ab = \pi R^2/(1-\epsilon^2)^{3/2} = \pi R^{1/2}a^{3/2}$.

8. One can determine the orbit completely from GM, L/m, and $2E/m$.

9. If we know any two of r_c, r_f, v_c, and v_f for an elliptical orbit, we can determine the other two using conservation of L and E:
   $L/m = r_c v_c = r_f v_f$;
   $2E/m = v_c^2 - 2GM/r_c = v_f^2 - 2GM/r_f$.

10. In solving orbit problems, it is useful to bring these equations in unitless form using $u = v_f/v_c$ and $q = r_f/r_c$:
$1 = uq$;
$1 - b = u^2 - b/q$, with $b = 2GM/(r_c v_c^2)$.