Physics 834: Problem Set #6

These problems are due in Weishi (Shirley) Li’s mailbox in the main office by noon on Thursday, November 3. Check the 834 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone; if you do these correctly you will get 100% of the points for the problem set. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (20 pts) A single-loop series LRC circuit has resistance $R = 15 \, \Omega$, inductance $L = 10 \, \text{mH}$, and capacitance $C = 1.5 \, \mu\text{F}$. A half-rectified sine wave power supply (see Problem 4.12 figure in Lea) with period $T = 1.57 \times 10^{-3} \, \text{s}$ is attached to the circuit. Find the voltage across the capacitor as a Fourier series in time once the circuit has reached a steady state.

2. (20 pts) Show using contour integration that the following sequences of functions are delta sequences:
   (a) $\phi_n(x) = \frac{n}{\pi} \left( \frac{1}{1 + n^2 x^2} \right)$
   (b) $\phi_n(x) = \frac{1 - \cos n \pi x}{n \pi x^2}$

3. (20 pts) A uniform rod of length $l$ and mass $M$ lies along the $x$-axis with one end at the origin. Express the density in terms of delta functions
   (a) in rectangular Cartesian coordinates
   (b) in cylindrical coordinates
   (c) in spherical coordinates

4. (20 pts) Evaluate
   (a) $\int_{-\infty}^{\infty} e^{-|x|} \delta(x^2 + 2x - 3) \, dx$
   (b) $\int_{-\infty}^{\infty} e^{-x^2} \delta(x^2 + x - 6) \, dx$

5. (20 pts) A string of length $L$, with tension $T$ and mass per unit length $\mu$, is hit simultaneously at $t = 0$ at the two points $x = L/3$ and $x = 2L/3$. The impulse delivered at each point is $I$. Find the subsequent displacement of the string.

Optional problems (counts as bonus points)

6. (10 pts) Which series, the sine series or the cosine series, do you expect will converge more rapidly to the function $f(x) = x^3$ on the range $0 < x < 1$? Give reasons for your answer. Evaluate the first four nonzero terms in the optimum series. How large is the fraction deviation $\frac{|S_4 - f(x)|}{f(x)}$ at $x = 0.5$ and $x = 1$? In this expression, $S_4$ is the sum of the first four terms.
7. (10 pts) Find a Fourier series representation of the delta function \( \delta(x) \) in the range \((-L, +L)\) in two ways:

(a) Start with the Fourier series for a step function and differentiate.

(b) Start with the block functions (which form a delta sequence)

\[
\phi_n(x) = \begin{cases} 
  n/2 & \text{if } -1/n < x < 1/n \\
  0 & \text{otherwise}
\end{cases}
\]

(1)

and form the Fourier series. Take the limit as \( n \to \infty \).

(c) Are the results the same? If not, why not? Give a quantitative as well as a qualitative account of any discrepancy.

8. (10 pts) Prove the relation

\[
\nabla^2 \ln \frac{\rho}{a} = 2\pi\delta(\vec{\rho})
\]

(2)

where \( \rho \) is the radial coordinate in a cylindrical coordinate system, \( a \) is a constant length, and \( \vec{\rho} \) is the position vector in a plane. Use the result to find the potential due to a line charge \( \lambda \) running parallel to the \( z \)-axis at \( x = a, y = b \).