Physics 834: Problem Set #4

These problems are relevant for the midterm, but will not be graded. They are not to be handed in! Solutions will be available two days before the midterm. Practice problems for the midterm will be available later this week. Check the 834 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

Differential equation problems

1. (10 pts) Determine a solution of the equation

\[(1 + x)y'' + (3 + 2x)y' + (2 + x)y = 0 \tag{1}\]

at large \(x\). Use this to determine the solutions for all \(x\). (Hint: you should find a second-order equation with constant coefficients, which you can solve with a \(e^{sx}\) ansatz. The remaining differential equation can be solved by direct integration.)

2. (10 pts) Attempt to solve the equation

\[x^2y'' + y' = 0 \tag{2}\]

using the Frobenius method. Show that the resulting series does not converge for any value of \(x\) (you may need to look up tests of convergence of a series).

3. (10 pts) The Schrödinger equation in one dimension has the form

\[\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + (E - V)\psi = 0 \tag{3}\]

Develop a series solution for \(\psi\) in the case where \(V\) is the potential due to the interaction of two nucleons:

\[V(x) = C e^{-ax} x \tag{4}\]

Assume \(\psi(0) = 0\). Obtain at least the first three nonzero terms.

4. (10 pts) Weber’s equation is

\[y'' + \left(n + 1 - \frac{x^2}{4}\right) y = 0 \tag{5}\]

where \(n\) is an integer. Show that the substitution \(y = \exp(-ax^2)v(x)\) simplifies this equation for a choice of \(a\) that you are to determine. Find two solutions for \(v(x)\) as power series in \(x\).

5. (10 pts) The Stark effect describes the energy shift of atomic energy levels due to applied electric fields. The differential equation describing this effect may be written

\[xy'' + y' + \left(k - \frac{1}{4}Ex^2 + \frac{Ux}{2} - \frac{m^2}{4x}\right) y = 0 \tag{6}\]

where the term \(Ex^2/4\) is the perturbation due to the electric field. Obtain a power series solution for \(y\) and obtain explicit expressions for the first four nonzero terms. How many terms are needed before any effect of the electric field is included?
6. (10 pts) Suppose we are solving a problem in which we have a quantum mechanical potential \( V(x) \) in one dimension, but we only know it numerically on a uniformly spaced set of mesh points: \( x = 0, h, 2h, \ldots, Nh \) (so there are \( N + 1 \) points). We need to solve the bound state \((E < 0)\) Schrödinger equation with this potential:

\[
\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + [E - V(x)]\psi = 0
\]

(7)

for a given value of \( E \) (later we’ll discuss how to determine \( E \)) using a numerical method such as Runge-Kutta. To do this we will integrate out from \( x = 0 \) and in from \( x = Nh \) (this is numerically stable). To start these integrations using the Numerov method (see the article by Blatt), we need initial values for the wave functions at small and large \( x \).

(a) Suppose we choose \( \psi(0) = C_0 \), where \( C_0 \) is a constant to be determined later. Use the series method to provide a two-term starting approximation for \( \psi(h) \), under the assumption that \( V(x) \approx V_0 < 0 \), a constant, for small \( x \).

(b) For integrating in, find starting approximations for \( \psi(Nh) \) and \( \psi((N-1)h) \) Find a starting approximation to \( \psi'(Nh) \), under the assumption that \( V(x) \) goes to zero rapidly before \( x = Nh \). Assume a normalization factor \( C_\infty \), with \( C_\infty \) a constant to be determined later.

(c) (Bonus) Refine your starting values for \( \psi(x) \) to account for the slow but non-zero variation of \( V(x) \) near the origin and the small but not exactly zero value of \( V(x) \) for large \( x \).