Physics 834: Problem Set #3

These problems are due in Dr. Vladimir Prigodin’s mailbox in the main office by 4pm on Wednesday, October 12. Check the 834 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone; if you do these correctly you will get 100% of the points for the problem set. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (20 pts) Evaluate the following integrals using contour integration. Be sure to include all parts of the contour and if you set some part(s) to zero, give a justification. Verify each of your answers with Mathematica, giving the command you used (preferably on a separate printed sheet but writing it by hand is acceptable).

   (a) \( \int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}} \, dx \) where \( b \) is real and \( 0 < \text{Re}(a) < b \). Use a rectangular contour.

   (b) \( \int_{0}^{+\infty} \frac{x^{1/3}}{x^2+1} \, dx \) (do this without changing variables such as \( x = y^3 \))

2. (10 pts) The unit step (\( \theta \)) function is defined for real \( a \) as

   \[
   \theta(t-a) = \begin{cases} 
   0 , & t < a \\
   1 , & t > a 
   \end{cases} \tag{1}
   \]

   Show that \( \theta(t) \) has the integral representations

   (a) \( \theta(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ikt}}{k-i\epsilon} \, dk \)

   (b) \( \theta(t) = \frac{1}{2} + \frac{1}{2\pi i} \mathcal{P} \int_{-\infty}^{+\infty} \frac{e^{ikt}}{k} \, dk \)

3. (10 pts) Show that

   \[
   \int_{0}^{\infty} \sin(x^2) \, dx = \int_{0}^{\infty} \cos(x^2) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \tag{2}
   \]

   using the contour in Arfken Fig. 7.15. (This result has numerous applications in physics—for example, in signal propagation.)

4. (10 pts) Use the series method to find a solution of Laguerre’s differential equation

   \[
   xy'' + (1-x)y' + \alpha y = 0 \tag{3}
   \]

   that is regular at the origin. Show that if \( \alpha \) is an integer \( k \), then this solution is a polynomial of degree \( k \).
5. (10 pts) Solve the Bessel equation

\[ 4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \]  

as a Frobenius series in powers of \( x \). Sum the series to obtain close-form expressions for the two solutions.

Optional problems (counts as bonus points)

6. (10 pts) Evaluate the following integrals using contour integration. Be sure to include all parts of the contour and if you set some part(s) to zero, give a justification. **Verify each of your answers with Mathematica, giving the command you used (preferably on a separate printed sheet but writing it by hand is acceptable).**

   \begin{align*}
   (a) \quad & \int_{0}^{\pi} \frac{1}{1 + \cos^2 \theta} \, d\theta \\
   (b) \quad & \int_{0}^{+\infty} x^2 \cosh ax \, dx \text{ for real } a.
   \end{align*}

7. (5 pts) Use the calculus of residues to prove the identity:

\[ \int_{0}^{\pi} d\theta \cos^{2n} (\theta) = \pi \frac{(2n)!}{2^{2n}(n!)^2} = \pi \frac{(2n - 1)!!}{(2n)!!} \quad n = 0, 1, 2, \ldots \]  

(Note: the double factorial is defined in Arfken 8.1.)

8. (5 pts) In the quantum theory of atomic collisions we encounter the integral:

\[ I = \int_{-\infty}^{+\infty} \frac{\sin t}{t} e^{ipt} \, dt \]  

with \( p \) real. Show that:

\[ I = \begin{cases} 
0 , & |p| > 1 ; \\
\pi , & |p| < 1 .
\end{cases} \]

What happens if \( p = \pm 1 \)?