Lecture plan:

- midterm comments and example problems
- Solving $S$ eqn for bound states (Numerov method) [68-70]
- Fourier series introduction [71]
- Fourier series examples (with Mathematica integrals and plots)
- Gamma function follow-ups

Before class:

- Start up IE with 834 page. Start Mathematica with FourierSeries.nb.
- Hand back PS#3 (if available)
- Try starting up Naps Fourier transforms (cupsw0.zip) (put on this if too awkward)

On board:

- Comments on PS#3
- Midterm practice warm-ups
- Gamma function follow-ups
Midterm basic problems: Examples

What is?
\[ \varepsilon_{abc} \varepsilon_{def} = [\varepsilon_{abc} \varepsilon_{def}, \text{no repeated indices!}] \]
\[ \varepsilon_{abc} \varepsilon_{def} = [\delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}] \text{ standard identity} \]
\[ \varepsilon_{abc} \varepsilon_{cab} = [\varepsilon_{cde} = 0 \text{ because } \varepsilon_{ijk} = 0 \text{ for repeated indices}] \]

Show
\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \]
assuming these commute
\[ = A_a (B_x C_y - B_y C_x) = \varepsilon_{abc} A_a B_b C_c = \varepsilon_{abc} C_a A_b B_c = \varepsilon_{abc} A_a B_b C_c = C \cdot (\mathbf{A} \times \mathbf{B}) \quad \text{QED} \]

\[ \nabla \times (\nabla \phi) = 0 \Rightarrow (\nabla \times \nabla \phi)_a = \varepsilon_{abc} \frac{\partial}{\partial x^b} \frac{\partial}{\partial x^c} \phi = 0 \quad \text{(or } \varepsilon_{abc} \varepsilon_{def} \phi \text{ symmetric }) \]

Find \( \nabla \times \mathbf{X} \) in cylindrical coordinates
\[ \mathbf{X} = \hat{r} \rho \phi, \quad \text{or} \quad \mathbf{A}_1 = \hat{r}, \mathbf{A}_2 = 0, \mathbf{A}_3 = z \Rightarrow \nabla \times \mathbf{X} = \hat{r} \frac{\partial}{\partial \rho} (\rho \phi) + \hat{z} \frac{\partial}{\partial z} = 3 \hat{z} \quad \text{V} \]

Find \( \nabla \times \hat{\theta} \) in spherical coordinates
\[ \mathbf{A}_1 = 0, \mathbf{A}_2 = 1, \mathbf{A}_3 = 0 \Rightarrow \nabla \times \hat{\theta} = \hat{\phi} \frac{\partial}{\partial \phi} (\sin \theta) = \sin \theta \quad \text{V} \quad \text{(all others zero)} \]

Spot plz Error!
\[ \nabla \times (\mathbf{B} \times \mathbf{C}) = 0 \quad \text{[\( \mathbf{B} \times \mathbf{C} \) is a scalar, not a vector as need for cross products]}, \]
\[ (\mathbf{A} \times \mathbf{B})_a + (\mathbf{C} \times \mathbf{D})_a = \varepsilon_{abc} A_b B_c + \varepsilon_{def} C_d D_e \quad \text{[the two vectors must have B same index rather than 0 and d]} \]

What are the singularities of \( \frac{\sqrt{z^2 + 2z + 1}}{\sin z} \) and what type are they?
\[ z^2 + 2z + 1 < (z + 1)^2 \Rightarrow \sqrt{z^2 + 2z + 1} \text{ have branch points at } z = -2 \text{ and } z = -1 \]
\[ \sin z \text{ has zeros on the } x \text{ axis for } y = 0, x = \pi n \text{, which mean simple poles } \]

\[ \]
Would you close in the upper or lower half plane when doing the contour integral for \( \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \)? [Either! For \( z=Re^{it} \) the integral goes to zero as \( R/2^a \to R \) for \( R \to \infty \) in either half plane.]

What contour would you choose for \( \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \)?

[Split into \( (\frac{e^{it}}{x} - \frac{e^{-it}}{x})/2i \) and define each with a principal value. Jordan's lemma says we can close the first in the upper half plane and the second in the lower half plane, and the integral over the large semi-circle vanishes in each case.]

Find the residue of:

a) \( \frac{e^{1/2}}{z(z-i)} \) at \( z=0 \), \( z=i \):
\[
\text{Res}(0) = \frac{1}{(z-i)} = \frac{1}{ia}, \\
\text{Res}(i) = e^{i/2}/ia.
\]

b) \( \frac{z^2}{z^4-1} \) at \( z=i \):
\[
\text{Use } \text{Res}(i) = \frac{g(i)}{h''(i)} = \frac{2i}{4i^2} = -\frac{1}{2}.
\]

For differential equations, be prepared to:

a) apply the Frobenius method to derive a recurrence relation (e.g. \( a_n a_{n+2} = \text{odd} \)) and find the first few terms explicitly.

b) find the asymptotic behavior of \( y(x) \), as in class and P3#4.
Gamma Function Recap and Follow-Ups

Definitions (see Artken chap. 8)

\[ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \quad \text{Re} \, z > 0 \quad \text{[Why? Integral diverges otherwise]} \]

Other forms

\[ \Gamma(z) = 2 \int_0^\infty e^{-t} t^{z-1} \, dt \quad \text{Re} \, z > 0 \]

\[ \Gamma(z) = \sum_{n=1}^{\infty} [\ln(n!)^z - 1] \, dt \quad \text{Re} \, z > 0 \]

\[ \Gamma(z) \equiv \lim_{n \to \infty} \frac{1 \cdot 2 \cdot 3 \ldots n}{z(z+1)(z+2)\ldots(z+n)} \quad z \neq 0, -1, -2, \ldots \]

(artken proves equivalence)

Contour representation: \[ \int_C e^{z^+} \, dz = (e^{-1}) \Gamma(\nu+1) \]

Use \( \Gamma(z+1) = z \Gamma(z) \) to continue \( \Gamma \) integral forms to all \( z \) (except simple poles at \( z = 0, -1, -2, \ldots \)).

Keep in mind the form of \( \Gamma(z) \) will arise from simple transformations of integrals of interest:

\[ \int_0^\infty e^{-x^4} \, dx = \int_0^\infty e^{-u} \, du = \Gamma(\frac{1}{4}) \]

\[ \int_0^1 x^k \ln x \, dx = \int_0^1 (e^{-u})^k (1-u) \, du = -\int_0^\infty e^{-(k+1)u} \, du + \frac{1}{(k+1)^2} \int_0^\infty e^{-u} \, du \]

\[ \int_0^\infty e^{-x} \, dx = \int_0^\infty e^{-u} \, du \]

\[ \int_0^\infty x^k \ln x \, dx = \int_0^\infty (\ln x)^k \, dx \]

\[ \int_0^\infty x^k \ln x \, dx = \int_0^\infty (\ln x)^k \, dx \]
Fourier Series: Pass 2

Consider the expansion of the function $f(x)$ first in the interval $0 \leq x \leq \pi$ and then generalized to $0 \leq x \leq L$ or $-L \leq x \leq L$ (simple change of variable): [This is Lea's notation]

$$f(x) = \sum_{n=0}^{\infty} \left( a_n \sin nx + b_n \cos nx \right) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Unfortunately, Arken uses a different notation:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)$$

Splitting out the constant term explicitly, and with $a_0$ so that the formula below for $a_n$ applies to all.

$\Rightarrow$ be careful when using formulas from the books!

The function we want to Fourier analyze does not have to be periodic. If it is not, our Fourier series will only be a good representation in $[0, \pi]$.

- The conditions on $f(x)$ sufficient for an expansion to exist is that it is "piecewise regular": only a finite # of finite discontinuities and a finite number of maxima and minima in $[0, \pi]$.

- These hold for most physical problems.

- The ability of a Fourier series to represent discontinuities is in contrast to Taylor series.

- Given a Fourier series, we can integrate term-by-term but be careful differentiating — it may no longer converge.

Think of a Fourier series expansion in the larger context of orthogonal expansions: $f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$ for a complete basis $\{\phi_n\}$.

More abstractly:

$$\langle f, \phi_n \rangle = \int f(x) \phi_n(x) dx$$
In the present case, the $<\phi_m|f>$ integrals are:

\[
\int_0^{\pi} (\sin nx)(\sin mx) \, dx = \pi \delta_{nm} \quad \text{if } n \neq m
\]

\[
\int_0^{\pi} (\cos nx)(\cos mx) \, dx = \pi \delta_{nm}
\]

\[
\int_0^{\pi} (\sin nx)(\cos mx) \, dx = 0
\]

- See the Mathematica notebook `Fourier_series.nb` to verify these results. Other ways to do these integrals:
  - Convert to exponentials [e.g., $\sin x = (e^{ix} - e^{-ix})/2i$] and integrate directly.
  - Use trig identities to reduce to single sine or cosine, e.g., $\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

- Check the same plots (and do others) so the intuition of why they are orthogonal becomes clear.

Observation: The coefficients of the Fourier series are like components of the vector $f$? in a particular basis, so naturally we isolate the coefficient by the dot product $<\phi_m|f>$.

We will see many more examples of basis functions later.
If we can exchange the order of summation and integration,

\[ a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(mx) \, dx \quad m \neq 0 \]

\[ b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(mx) \, dx \quad m \geq 1 \]

\[ b_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \quad \text{[in Arfken this term is } \frac{1}{2}a_0 \text{]} \]

For a complex series,

\[ c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx \quad \text{[same idea to project]} \]

\[ \text{If } f(x) \text{ is real, } c_n^* = c_{-n} \text{ holds.} \]

If we switch to \( x \in [0, L] \), then

\[ a_m = \frac{L}{\pi} \int_0^L f(x) \sin(n \frac{2\pi x}{L}) \, dx \]

\[ b_m = \frac{L}{\pi} \int_0^L f(x) \cos(n \frac{2\pi x}{L}) \, dx \]

\[ b_0 = \frac{1}{L} \int_0^L f(x) \, dx \quad \text{(average of } f(x)) \]

- normalization constant for \( a_m, b_m \) is \( 2L \) (length & interval)
- argument of sines, cosines is \( k_n x \) where \( k_n = n \frac{2\pi}{L} = 2\pi \frac{n}{L} \), \( n \in \mathbb{Z} \)

For a complex series \( f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i n \frac{2\pi x}{L}} \)

\[ \Rightarrow c_n = \frac{1}{L} \int_0^L f(x) e^{-i n \frac{2\pi x}{L}} \, dx \]
Apply to a step function: \( \Theta(\frac{1}{2} - x) \) in \([0, 1] \):

![Graph of \( \Theta(\frac{1}{2} - x) \) in \([0, 1] \)]

Check the following in the Mathematica Fourier series notebook:

\[
\begin{align*}
\alpha_n &= \frac{2}{\pi} \int_0^\frac{\pi}{2} \Theta(\frac{1}{2} - x) \sin(n \pi x) \, dx
= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n \pi \frac{1}{2})}{n} \\
&= \left\{ \begin{array}{l}
0 \quad \text{if } n \text{ even} \\
\frac{2}{\pi n} \quad \text{if } n \text{ odd}
\end{array} \right.
\end{align*}
\]

\[
\beta_n = 2 \int_0^{\frac{\pi}{2}} \cos(n \pi x) \, dx = 0
\]

[These are all even about \( x = \frac{1}{2} \), but \( \beta_n \) is odd \( \Rightarrow \) vanish.]

\[
\beta_0 = 1 \int_0^{\frac{\pi}{2}} 1 \, dx = \frac{\pi}{2} \quad \Rightarrow \quad \text{Pure is a constant offset}
\]

\[
\Rightarrow \quad \frac{\pi}{2} \xrightarrow{\frac{1}{2}} x
\]

is expanded in sines. Note what happens at \( x = 0, \frac{1}{2}, 1 \):

\[
\Rightarrow \quad f(x) = \Theta(\frac{1}{2} - x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{-1}{\pi n} \right) \sin(n \pi x)
\]

\[
= \frac{1}{2} - \sum_{m=0}^{\infty} \frac{\sin(2m\pi x)}{2m+1}
\]

Look at Mathematica notebook to see how the function is built up.

Try \( n_{\max} = 4, 5, 10, 50, \ldots \)

Where there is a discontinuity in the function, the Fourier series evaluates to the midpoint of the jump \( \Rightarrow \frac{1}{2} \) at \( x = 0, \frac{1}{2}, 1 \).

Look at Gibbs overshoot \( \Rightarrow \) value as peak is 1.179 and doesn't improve from the discontinuity \( \Rightarrow \) be careful in your numerics in such places must be accurate.
We have two other examples in the Mathematica notebook to consider:

i) An inverted parabola $f(x) = 1-x^2$ in $[-1,1]$

ii) A half circle $f(x) = \sqrt{1-x^2}$ in $[-1,1].$

For these examples,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$$

$$b_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \quad \text{(average again)}$$

Notice the patterns of convergence.

⇒ next time consider the nature of the convergence of the series toward $f(x).$

- Depending on whether $f(x)$ is even or odd, we can consider separate sine and cosine series.

The real issue is the periodic extension by the expansion outside the interval. See the discussion in [Ref].

Parseval's theorem says how the integral of the square of $f(x)$ is related to the sum of squares of coefficients. E.g.,

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

⇒ $\langle f | f \rangle = \sum_{n} \langle f | \phi_n \rangle \langle \phi_n | f \rangle = \sum_{n} |c_n|^2$
Using Fourier Series to Solve Diff. Eqs.

We'll consider two representative examples.

A. An inhomogeneous linear equation with a periodic driving term (periodic but not simply sinusoidal at one frequency)

B. Solution to a wave equation with fixed ends at x=0, L and given an initial condition at t=0.

Examples of A. are a damped, driven, harmonic oscillator:

\[ \frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + k^2x = f(t) \]

or for some system in electric circuit form: An RLC circuit

\[ \begin{align*}
I & \Rightarrow \quad L \frac{dI}{dt} + RI + \frac{Q}{C} = E(t) \quad \text{(by voltage drops)} \\
(\text{It} & \text{ is the charge}, \quad \text{It} & \text{ is the current})
\end{align*} \]

With \( I = \frac{dQ}{dt} \Rightarrow \frac{d^2Q}{dt^2} + \left( \frac{R}{L} \right) \frac{dQ}{dt} + \left( \frac{1}{LC} \right) E(t) = \dot{Q} + 2\omega Q + \omega^2 Q \]

First consider if \( E(t) \) has a single frequency, we could take \( E(t) = E_0 \sin(\omega t) \), but this is awkward with both first and second derivatives \( \Rightarrow \) use exponentials \( \Rightarrow \) \( E(t) = E_0 e^{i\omega t} \).

The equation is linear, so we expect the response to be at the same frequency in steady state \( \Rightarrow \) \( \text{solution to the inhomogeneous equation} \)

General solution includes

1. Linear is critical; if \( Q^2 \) appears, then not all terms have the same exponential \( \Rightarrow \) can't factor \( e^{i\omega t} \) out.
But if it is linear, then we can solve for a periodic but otherwise general \( E(t) \) by Fourier decomposing \( E(t) \) into frequency components:

\[
E(t) = \sum_{n=-\infty}^{\infty} E_n e^{i\omega_n t} \quad \text{where} \quad \omega_n = \frac{2\pi n}{T}, \quad T = \text{period}
\]

So the initial problem is to find \( E_n \) given the form of \( E(t) \).

E.g., a square wave, as in Lea's example:

\[
E(t) = E_0 \left( \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{i}{n} e^{i\omega_n t} \right)
\]

Then we take

\[
Q(t) = \sum_{n=-\infty}^{\infty} q_n e^{i\omega_n t}
\]

Substitute into the diff. eq. and equate terms with the same \( \omega_n \).

In detail, we project the \( m \)th term by multiplying the full equation by \( e^{-i\omega_m t} \) and integrating over \( t \) for 0 to \( T \)

\[\Rightarrow \text{only } n=m \text{ survives:}\]

\[
\sum_{n=-\infty}^{\infty} \left( -\omega_m^2 + 2\alpha \omega_m + \omega_m^2 \right) q_n = E_m \Rightarrow \text{Solve for } q_m
\]

\[
q_m = \frac{E_m}{\omega_m^2 - \omega_m^2 + 2\alpha \omega_m} = \frac{E_m}{(\omega_m^2 - \omega_m^2 + 2\alpha \omega_m)} = \frac{E_m}{(\omega_m^2 - \omega_m^2 + 2\alpha \omega_m)}
\]

\< resonance when \( \omega_m \approx \omega \)

See Lea for the square wave solution, when \( \omega \) and \( \alpha \) are inserted in the \( q_m \) equation, we get real quantities.
For problems of type B we consider the wave equation for a string attached (fixed ends) at \( x = 0 \) and \( x = L \). At \( t = 0 \) we pluck it in the middle and let it go. This establishes the initial conditions \( y(x, t=0) \) and \( y_x(x, 0) = 0 \).

For example:

The wave equation is \( \frac{1}{v^2} \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \) when the wave velocity \( v \) in this case depends on the mass density and the tension.

Goal: find \( y(x, t) \) for all \( t \) and \( 0 < x < L \).

Plan: Apply separation of variables and then insert a general Fourier series for the time dependence after Fourier analyzing \( y(x, 0) \).

Separation of variables is based on the observation that the conserved \( y(x, t) = A(x) B(t) \) results in two separated equations for the \( x \) and \( t \) dependence.

\[
\frac{v^2}{A} \frac{\partial^2 A}{\partial x^2} = \frac{\partial^2 B}{\partial t^2} \Rightarrow \frac{\partial^2 A}{\partial x^2} = \frac{\partial^2 B}{\partial t^2} \Rightarrow \frac{\partial^2 A}{\partial x^2} = \frac{\partial}{\partial t} \left( \frac{\partial^2 B}{\partial t} \right)
\]

Key: \( \frac{A''}{A} \) only depends on \( x \) while \( \frac{B''}{B} \) only depends on \( t \), so each separately must be a constant independent of \( x \) and \( t \).

\[
\Rightarrow \frac{A''}{A} = -k^2 \Rightarrow A = \sin(kx) \text{ with } k = k_n = \frac{n\pi}{L} \text{ to vanish at } x = 0, x = L.
\]

Then \( \frac{B''}{B} = -k^2 \Rightarrow -k^2 \frac{\partial^2 B}{\partial t^2} \Rightarrow \sin(wt) \cos(wnt) \) are solutions. \( \Rightarrow wn = \frac{n\pi}{L} \sqrt{\frac{v^2}{L}} \)

\( \Rightarrow \) general solution \( y(x, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{L} x \right) \left[ a_n \sin \left( \frac{n\pi}{L} wt \right) + b_n \cos \left( \frac{n\pi}{L} wt \right) \right] \)

Boundary conditions fix this.
Note that $a_n$, $b_n$ absorb any overall coefficient that we might have put with $\sin(kx)$. The coefficients $a_n$, $b_n$ are determined completely by the initial conditions at $t=0$.

- Pluck from rest $\Rightarrow \frac{dy}{dt}|_{t=0} = 0 \Rightarrow a_n = 0$ for all $n$.
- Find $b_n$ from: $y(x,0) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{nx}{L} \right)$

(same Fourier decomposition problem considered earlier!)

- Suppose we start with a simple half sine wave: $\sin \left( \frac{nx}{L} \right)$. We expect $\cos(ut)$, with $u = N$ or $u = kN$
- It stays a sine wave, with $\cos(ut)$ modulated amplitude.

Now example 4.4 in Lea.

See the Mathematica notebook Fourier SineSeries.nb for the calculation of the coefficients.

Full solution:

$$ y(x,t) = \frac{6h}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi vt}{L} \right) \quad n \text{ odd} $$

- only harmonics even about the middle of the string.
- Note: we could put the origin in the middle. Might be more convenient, but it is not necessary to solve the problem.