Before class:

- (If available) return graded PS#2. Avg. 86 (contact 90)
- Start up 834 web page and Mathematica but don’t project
- Put principal value recap and Jordan’s Lemma “Spot the Error” on the board
- Also on board: Schedule for next two weeks
  - No class Wednesday, PS#3 due at 4pm Wednesday?
  - Make-up course on Friday in Smith 1004
    - Thursday 5pm for those with TA duties or out-of-town for APS meeting
  - Midterm on Wednesday, Oct. 19 in class on vector calculus, complex analysis, differential equations by series + asymptotic solutions
  - Short(!!) PS#4 will be on remaining diff. eq. topics
  - Fourier series are next
  - "Spot the Error!" Jordan’s Lemma edition

Discussion:

Most important aspects of core competencies:

- Basic complex variables, manipulations (e.g., solutions to equations), complex analysis, mostly series and asymptotic solutions

Comments on homework:

- New policy: Identify required problems — these will be the foundation for everyone and the basis for grades (do well on these problems and exams and you get an A; these also determine minimum core competency for passing).
- Retroactive PS#2 scoring: score is out of 90 — additional is bonus
- Bonus can only help; not drag down anyone
- Use Mathematica to do (or check) algebra
- For major losses of points, see me to arrange make up.

Lecture plan: "Spot the Error": Principal value and dispersion relations; differential equations; if time: Further example, PS#4 topics
"Spot the Error!" — Jordan's lemma edition

Consider \( g(z) = e^{iz} \) with \( k>0 \), \( \lim_{|z| \to \infty} g(z) = 0 \) on the contour \( \gamma \) as \( \arg(z) \to 0^+ \).

On \( \gamma \), \( |z| = R \), \( dz = iRe^{i\theta} \, d\theta \)

\[ \Rightarrow \int_{\gamma} g(z) e^{iz} \, dz = \int_0^{2\pi} f(Re^{i\theta}) e^{iR \cos \theta} \, e^{-R \sin \theta} \, d\theta \]

Claim: As \( R \to \infty \), \( e^{-R \sin \theta} \) dominates \( R \) and everything else so the integral \( \to 0 \).

Why is this claim incorrect?

It \( \neq e^{|z|} \) for different \( R \)

So \( e^{-R \sin \theta} \to 0 \) except in a region \( \sim \frac{1}{R} \theta \)

and \( \frac{\theta}{R} \) with \( \frac{1}{R} \to \text{the contribution is } O(1) \) that only just cancels \( \frac{R}{\theta} \) from the measure. \( \Rightarrow \) we need \( f(Re^{i\theta}) \to 0 \)

to justify Jordan's lemma. Note: Jordan's lemma with \( k>0 \)
and closing in the lower half plane works just as well due to symmetry.

- Do we have this issue with rectangular contours? No.

So \( e^{-R \sin \theta} \) on the right is determined by \( e^{-aR} \) and on the left by \( e^{-aR} \).

- What about \( \int g z^z \, dz \) on a curved path? \( \text{Yes, } \int g \text{ over } e^z \).

Now \( z = Re^{i\theta} \to \int_0^{2\pi} e^{R^2 \sin \theta} e^{-R \cos \theta} e^{iR \theta} \, d\theta \)

and \( \int e^{-R^2 \sin \theta} \) term is now \( O(\frac{1}{R^2}) \) in the integral so it still wins over \( \int g \).
Principal value follow-up...

If we integrate a function with a pole on the real axis, \( g(x) = \frac{f(x)}{x-x_0} \), the principal value integral

is defined as

\[ P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} \, dx = \lim_{\varepsilon \to 0} \left[ \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{f(x)}{x-x_0} \, dx + \int_{x_0+\varepsilon}^{\infty} \frac{f(x)}{x-x_0} \, dx \right] \]

So we can calculate the principal value directly from this definition.

Or, we can relate it to two ways to calculate a contour integral:

\[ \int_{\gamma} f(z) \, dz = P \int_{\gamma} \frac{f(z)}{z-z_0} \, dz + \lim_{\varepsilon \to 0} \int_{\gamma_{\varepsilon}} \frac{f(z)}{z-z_0} \, dz \]

The integrals over the semi-circles go to zero.

Symbolically:

\[ \frac{1}{x-x_0+i\varepsilon} = \frac{P}{x-x_0} + \pi i \delta(x-x_0) \]

Similarly:

\[ \int_{-\infty}^{0} \frac{f(x)}{x-x_0} \, dx = P \int_{-\infty}^{0} \frac{f(x)}{x-x_0} \, dx + \lim_{\varepsilon \to 0} \int_{0}^{\infty} \frac{f(x+i\varepsilon)}{x-x_0} \, dx \]

\[ \frac{1}{x-x_0+i\varepsilon} = \frac{P}{x-x_0} - \pi i \delta(x-x_0) \]

Finally, \( \frac{P}{x-x_0} = \frac{1}{2} \left( \frac{1}{x-(x_0+i\varepsilon)} + \frac{1}{x-(x_0-i\varepsilon)} \right) \), which we've used already.
Further comments on differential equations:

- Lea lists some methods of solution
  1. Guess the form (e.g., undetermined coefficients)
  2. Power-series type solution (Frobenius) — this can be useful for numerical calculations as well
  3. Asymptotic solution
  4. Relate (possibly with change of variable) to known form (e.g., hypergeometric equation)
  5. Integrate numerically

Most of the rest of the course will touch on different features of differential equations!

- Sometimes Frobenius doesn't give us a good solution (indicial equation has repeated root or roots differ by an integer).
  - Problem is that there is a logarithm, and Frobenius doesn't allow for it.
  - So look for
    $$y_d = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+p}$$
    First solution $y_d(x)$

- If we want to expand around a singular point (e.g., $x=1$ for the Legendre equation), simply change to $w=x-1$ so that the singularity is at $w=0$, and apply Frobenius to get $y(w)$. See Lea Example 3.7.

- The indicial equation may have complex roots. Just do it!
Asymptotic methods ⇒ look at large values of $x$.
This is often helping in checking or starting numerical solutions.

Consider the example in Lee chapter 3, (example 3.9).
Modified Bessel equation is:
$$y'' + \frac{1}{x}y' - \left(1 + \frac{m^2}{x^2}\right)y = 0.$$ 

Find $y(x)$ as $x \to \infty$.
* For large $x$, the $\frac{1}{x}y'$ and $-\frac{m^2}{x^2}y$ term get small,
so $y_\infty$ satisfies
  $$y_\infty'' - y_\infty = 0$$
* We know the solutions: $y_\infty = (\text{const}) e^{\pm x}$

Now we can extract this dominant large $x$ behavior by writing
$$y(x) = V(x) y_\infty(x) = V(x) e^{\pm x}$$
$$y' = V'e^{\pm x} \mp Ve^{\pm x}, \quad y'' = V'e^{\pm x} \mp 2Ve^{\pm x} + Ve^{\mp x}$$

Substituting in the equation
$$\Rightarrow V'e^{\pm x} \mp 2Ve^{\pm x} + Ve^{\pm x} + \frac{1}{x} (V'e^{\pm x} \mp Ve^{\pm x}) -\left(1 + \frac{m^2}{x^2}\right)Ve^{\pm x} = 0$$

Simplify:
$$\Rightarrow V'' + \frac{\mp 2\alpha}{x} V' + \frac{1}{x} (\pm \alpha - m^2) V = 0$$

Again, look at large $x$ large, assuming $V = x^{\alpha}(1 + \text{corrections})$,
$$\Rightarrow \alpha (\alpha - 1)x^{\alpha - 2} + \alpha x^{\alpha - 2} \pm 2x^{\alpha - 2} + x^{\alpha - 1} - m^2 x^{\alpha - 3} = 0$$

Keep leading terms: $x^{\alpha - 1} \Rightarrow \pm 2\alpha x^{\alpha - 1} = 0 \Rightarrow \alpha = -\frac{1}{2}$

$$\Rightarrow y = \frac{e^x}{\sqrt{x}} \quad \text{or} \quad y = \frac{e^{-x}}{\sqrt{x}}$$
are the asymptotic forms.
Probenius Method Example

\[ y(x) = x^k \sum_{n=0}^{\infty} a_n x^n \]

(1) Legendre equation:
\[ (1-x^2) y'' - 2x y' + \alpha(x+1) y = 0 \]

Check singularities:
\[ y'' - \frac{2x}{1-x^2} y' + \frac{\alpha(x+1)}{1-x} y = 0 \Rightarrow \text{isolated singular points } x = \pm 1 \]

Expand about \( x = 0 \) (see Lea's Example 3.7 for an expansion about 4):
\[ y = \sum_{n=0}^{\infty} a_n x^n \]
\[ y' = \sum_{n=0}^{\infty} (n+1) a_n x^n \]
\[ y'' = \sum_{n=0}^{\infty} (n+2) a_n x^{n+1} \]

Separate terms with different powers of \( x \):
\[ \sum_{n=0}^{\infty} a_n x^n - 2 \sum_{n=0}^{\infty} (n+1) a_n x^n \]
\[ - \sum_{n=0}^{\infty} (n+2) a_n x^{n+1} + \sum_{n=0}^{\infty} \alpha(x+1) a_n x^n = 0 \]

Lowest powers of \( x \):
\[ x^{k-2} \sum_{n=0}^{\infty} a_{k-n-1} x^n = 0 \]

For general term, demand that coefficient of \( x^{k+j} \), \( j \geq 0 \) vanish.

Set \( n = 1^2 \) in first term and \( n = j \) in others and switch to sum over \( j \):
\[ (j+1+x)(j+1+x) a_{j+2} - (j+1)(j+1) a_j - 3(j+1) a_j + \alpha(x+1) a_j = 0 \]

For \( k = 0 \):
\[ (j+1)(j+1) a_{j+2} = [j(j+1)+\alpha(x+1)] a_j \]

\[ a_{j+2} = \frac{\alpha(x+1)}{3} a_j \]
\[ a_{j+1} = \frac{4}{3} a_j \]
\[ a_2 = \frac{3}{3} a_0 \]
\[ a_3 = \frac{4}{3} a_2 = \frac{4}{3} \cdot \frac{3}{3} a_0 = \frac{4}{3} a_0 \]
\[ a_4 = \frac{7}{9} a_3 = \frac{7}{9} \cdot \frac{4}{3} a_0 = \frac{7}{9} \cdot \frac{4}{3} \cdot \frac{3}{3} a_0 = \frac{7}{9} a_0 \]

\[ y(x) = a_0 \left[ 1 - \frac{\alpha(x+1)}{3} x + \frac{\alpha(x+1)(x+3)(x+2)}{3} x^2 - \frac{\alpha(x+1)(x+3)(x+2)(x+1)}{9} x^3 + \ldots \right] \]

Even series. The general term is easy to imagine (but might not be nice)!
Now we can develop an odd series starting with $a_1$ either from $k=0$ or $k=1$.

For $k=0$, we have the same relation between $a_{j+2}$ and $a_j$, but now we start with $a_3$: 

$$a_3 = \frac{2-\alpha(x+1)}{3,2} a_2, \quad a_5 = \frac{3,4-\alpha(x+1)}{5,4} a_3, \quad a_7 = \frac{5,6-\alpha(x+1)}{7,6} a_5, \ldots$$

$$\Rightarrow y(x) = a_2 \left[ x - \frac{(x-1)(x+2)}{3!} x^3 + \frac{(x-3)(x+1)(x+2)(x+4)}{5!} x^5 + \ldots \right]$$

For $k=1$, the recurrence is: $(j+3)(j+2) a_{j+2} = (j+1)(j+2) a_j + 2(j+1) - \alpha(x+1)] a_j$ or $a_{j+2} = \frac{(j+1)(j+2) - \alpha(x+1)}{(j+3)(j+2)} a_j$. Starting from $j=0$ gives the same series (remembering that $k=1$ means we start from $x$ and have odd powers).

Do these series converge? Not for $x>1$ for general $\alpha$. However, if $\alpha$ is an integer $l$, then the series truncates to a polynomial (with $a_0=1$, $a_1=0$)

- For $l=0$, $a_2=0$ and all higher $\Rightarrow y(x) = 1, \quad [p_0(x) = 1]$
- For $l=1$, $a_3 = \frac{\alpha}{3,2} \Rightarrow y(x) = x, \quad [p_1(x) = x^2]$
- For $l=2$, $a_4 = -\frac{2}{3} a_3 \Rightarrow y(x) = x - \frac{3}{2} x^3, \quad [p_2(x) = \frac{1}{3} x^3]$
- For $l=3$, $a_5 = -\frac{3}{2} a_4 \Rightarrow y(x) = x - \frac{15}{8} x^3, \quad [p_3(x) = \frac{5}{3} x^3]$

The polynomials in $[\bar{x}]^2$ are the Legendre polynomials, which we'll see again! They are normalized on $(-1,1)$ by $\int_{-1}^{1} p_i(x) p_j(x) dx = \delta_{ij}$. 

\[ \text{\large (6)} \]