Before class:
- Return remaining graded PS#2's.
- Start up 834 web page and Mathematica but don't project.
- Put contour integral plan on board.
- Note that PS#3 is out and due in Dr. Priporis's mailbox next week.
- Plan for next week: No class Wednesday? (Friday?)

On board:
- Warm-up problem: "in your head" residue. Find residues at poles of
  \[ f(z) = \frac{e^{ikz}}{(z-a)(z+ib)} \]
  \[ \text{Res } f(a) = \frac{e^{ika}}{(a-b)(a+ib)} \]
  \[ \text{Res } f(-b) = \frac{e^{ikb}}{(-b-a)(-b+ia)} = \frac{e^{ikb}}{aib} \]
- Lead's steps for evaluating an integral (use \( \int_{-\infty}^{\infty} e^{it} \, dt \) as an example)

1. Draw complex plane with contour \( C \) chosen to include integral of interest and
   not of contour do-able. Mark poles or
   other singularities, including branch points and cuts.
2. If there is a branch cut, "Autoform" Ru contour so it
   doesn't hit Ru cut.
3. Notes: Poles inside \( C \) ("enclosed") \( \Rightarrow \) here \( z_0 = ti \)
4. Evaluate \( \text{Residue of } f \) at each enclosed pole:
   \[ \frac{1}{z-z_0} = \frac{1}{z-ti} = \frac{1}{z+ib} \Rightarrow z = \frac{1}{\sqrt{t}} \]
   a. Mathematica b. a. from Laurent series
   c. \( \lim_{z \to a} \frac{1}{z-z_0} = \lim_{z \to a} \frac{1}{z-ti} \]
5. Evaluate \( f(z) \) at pole \( a \)
6. Apply the residue theorem \( \int_{C} f(z) \, dz = 2\pi i \sum \text{residues enclosed poles} \)
7. Evaluate other parts or show \( Re \) \( \text{c} \)

Vanish \( \int_{\gamma} \frac{1}{z+ib} \, dz = \int_{0}^{\pi} \frac{\cos t}{\cos \pi} \, dt \to 0 \)
10/5/11
Go through other contour pages 140, 141, 142 ⇒ integral examples from start with 3, 4, 5 and revisit others if there is time.

6 Integrals with poles on the real axis.

In our examples so far along the real axis, the poles have always been somewhere in the complex plane. What if they are on the axis?

We need to specify (usually) based on physics, how to go around the pole or how to move the pole.

Let's use the example from Lea, pg 143 \( \int_{-\infty}^{\infty} \frac{\sin bx}{x-2} \, dx \) with \( k \neq 0 \), but we'll move the pole.

Split into \( e^{ikx} \) and \( e^{-ikx} \) pieces to take advantage of semicircle.

\[
\int_{-\infty}^{\infty} \frac{\sin bx}{x-2} \, dx = \frac{1}{2i} \left( \int_{-\infty}^{\infty} \frac{e^{ikx}}{x-2} \, dx - \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x-2} \, dx \right)
\]

Limit in upper half plane so Jordan’s lemma applies.

When \( \epsilon \to 0^+ \), the semicircle contour becomes:

\[
\int_{\infty}^{\infty} \frac{\sin bx}{x-2} \, dx \to \lim_{\epsilon \to 0^+} \int_{\infty}^{\infty} \frac{e^{ikx}}{x-2} \, dx
\]

Limit in lower half plane
implied when using $\mathrm{ie}$ or $\mathrm{im}$ that
\[
\lim_{\varepsilon \to 0} \int_{C_R^{\varepsilon}} \frac{e^{ikz}}{z-(a+i\varepsilon)} \, dz = 0 \quad \text{(no poles enclosed)}
\]

\[
\int_I = \int_{C_R} \frac{e^{-ikx}}{x-(a-i\varepsilon)} \, dx \quad \text{clockwise contour}
\]

\[
\int_{C_R} = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x-(a-i\varepsilon)} \, dx + \int_{C_R} \frac{e^{-ikx}}{z-(a-i\varepsilon)} \, dz \quad \rightarrow 0 \quad \text{uniformly}
\]

\[
\int_{C_R} = \frac{2\pi i}{k} \left( e^{-ik2} - 1 \right) \quad \text{by residue theorem}
\]

\[
\int_{-\infty}^{\infty} \frac{\sin(kx)}{x-(a+\varepsilon i)} \, dx = \int_{-\infty}^{\infty} \frac{\sin(kx)}{x^2} \, dx \quad \text{Suppose we did}
\]

\[
\Rightarrow \text{pick up pole in upper half plane} \quad \frac{1}{2i} \left( e^{-ik} + e^{ik} \right)
\]

Define principal value integral:
\[
P \int_{-\infty}^{\infty} \frac{\sin(kx)}{x^2} \, dx = \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{-2\varepsilon} \frac{\sin(kx)}{x^2} \, dx + \int_{2\varepsilon}^{\infty} \frac{\sin(kx)}{x^2} \, dx \right)
\]

\[
= \frac{1}{2i} (\pi e^{-ik} + \pi e^{ik}) = \frac{\pi \cos(2k)}{2}
\]

(like integrating above and below to remove the piece near the pole.)

Example: on pS13.
Brief mention: Dispersion relations

You'll see these in EM as the Kaneer-Kronig relations, which relate the real and imaginary parts of the dielectric constant of a material. They're dispersive (index of refraction) and absorptive properties are related.

They show up in nonrelativistic and relativistic scattering.

Claim: Given \( f(z) \) analytic in the upper half plane with \( |z| \to \infty \),

\[
\text{Re}[f(z_0)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x - z_0} dx
\]

principal

\[
\text{Im}[f(z_0)] = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x - z_0} dx
\]

Consider Cauchy integral formula over:

\[
f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z_0} dx + \text{vanishing } \text{semi circle if } z_0 \text{ in upper half plane,}
\]

It in lower half plane, the integral is zero.

Now let \( z_0 \) approach the axis from above \( z_0 = x_0 + \epsilon \) and below \( z_0 = x_0 - \epsilon \)

and assume:

\[
\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0 - \epsilon} dx + \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0 + \epsilon} dx = \frac{1}{\pi} \left[ \text{Re} [f(x_0 + \epsilon)] - \text{Re} [f(x_0) + \epsilon] \right] = \pi i \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx
\]

\[
\Rightarrow \quad [f(x_0) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx]
\]

Now let \( f(x) = \text{Re}[f(x)] + i\text{Im}[f(x)] = u + iv \) and equate real and imaginary parts, and we're done! (The \( \frac{1}{\pi i} \) exchanges real \( \leftrightarrow \) imaginary on the right side.)

Aside: Mathematica Integrate has PrincipalValue -> True option
Brief mention: Analytic continuation (more later in course)

Footnote in Arfken, p. 434: If two analytic functions coincide (i.e., have the same value) in any region or on any line segment, then they are the same function — that is, in regions where they are both well defined, they'll give the same answer.

This enables us to extend functions to regions in \( \mathbb{C} \) beyond where they are originally defined.

One way to do this analytic continuation is by overlapping circles of convergence of Taylor series.

The circle of convergence \( C_1 \) is where the expansion of \( f(z) \) about zero converges. So \( f \) in \( C_1 \) is defined by its Taylor series but not outside. However, there is an expansion about \( C_2 \) that is good within \( C_2 \).

In the overlap region, \( f \) is uniquely defined, so it is the same function within both \( C_1 \) and \( C_2 \). We have "continued" the series from \( C_1 \) to \( C_2 \). And repeat.

Alternative methods for analytic continuation will be considered later.
Follow-up: If $u$ and $v$ are the real and imaginary parts of an analytic function $w(z)$, then they are harmonic; they satisfy $\nabla^2 u = 0$, $\nabla^2 v = 0$ - Laplace's equations (in 2 dimensions). Proof is simple from C-R equations:

\[
\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = -\frac{dv}{dx}
\]

Differentiate again:

\[
\frac{d}{dx} \left( \frac{du}{dx} \right) = \frac{d}{dx} \left( \frac{dv}{dy} \right) \quad \frac{d}{dy} \left( \frac{du}{dy} \right) = -\frac{d}{dx} \left( \frac{dv}{dy} \right)
\]

\[
\Rightarrow \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = 0 = \nabla^2 u \quad \text{(and similarly for } v) \]

An application to fluid flow is given in 2.4.1, but also relevant to E&M.
In PS#3 you are asked to find solutions to Laguerre's differential equation

\[ xy'' + (1-x)y' + ay = 0 \]

and the Bessel equation

\[ x^2y'' + xy' + (x^2 - 1)y = 0 \]

by using series. Notation: \( y = y(x) \), \( y' = \frac{dy}{dx} \), \( y'' = \frac{d^2y}{dx^2} \)

Recall the terminology (knowing these is a core competency)

- **Order:** highest derivative in the equation.
- **Linear or Nonlinear:** does \( y, y', y'', \ldots \) appear as more than first power? Here: no, so linear.
- \( \Rightarrow \) Jackson says he will consider linear equations only.

**Nonlinear example:** \( yy' = 2 \)

Ordinary or partial: Does \( y \) depend on more than one variable (e.g., \( x \) and \( t \)) with partial derivatives for each?

\( \Rightarrow \) partial diff eq or PDE, here, \( y(x) \) so ordinary.

- Above equations are ODEs: \( \frac{d^2y}{dx^2} = x^2 \frac{dy}{dx} \)

Homogeneous or inhomogeneous:

- Does each term depend on \( y \) or derivatives \( \Rightarrow \) homogeneous

\[ \frac{dy}{dt} + 2x \frac{dy}{dt} + y = 0 \] (damped harmonic oscillator)

- We'll come back to inhomogeneous (driving terms \( \Rightarrow \) forcing functions, etc.)

Are coefficients constant? (yes here)
11/15/11

Many methods to solve differential equations, including numerical methods that are very important.

Here we'll consider power series solutions — expand around a point.

Distinguish between expanding about a regular and singular point of the differential equation.

Solve for $y'' = f(x, y, y')$ [nothing but $y''$]

If homogeneous, then

$$y'' + p(x)y' + q(x)y = 0$$

Cases

i) If $p(x), q(x)$ finite at $x = x_0$, $x_0$ is ordinary point

ii) If either diverge, then singular:

- regular if $(x-x_0)p(x) \neq 0, (x-x_0)^2q(x)$ stay finite

Bessel equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

$$\Rightarrow y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

$$\Rightarrow p(x) = \frac{1}{x}, \quad q(x) = 1 - \frac{n^2}{x^2} \Rightarrow x = 0 \text{ is regular singularity and no other singular points for finite } x.$$  

- Almost always true in physics equations that we have no worse than a regular singular point

$$\Rightarrow \text{can do series expansion}$$
Basic principles & series solutions
- We can expand the desired solution(s) in a series
  - it may be a Laurent series or overall fractal powers
- We can take derivatives of the series term by term (why?)
- We can equate coefficients of equal powers after we plug in the series into our equation. (Why?)
  [Uniqueness of power series]

Solution about a regular point.
- Do an example where we know the answer by inspection:
  \[ \frac{\partial^2 y}{\partial x^2} + k^2 y = 0 \]  (Helmholtz or simple harmonic oscillator equation)
  \[ \Rightarrow \text{know } y = \sin kx \text{ or } \cos kx \text{ are solutions.} \]

- No singular points, so \( x = 0 \) is a regular point.
  Assume
  \[ y = \sum_{n=0}^{\infty} a_n x^n \]
  \[ y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \] (really starts at \( n = 1 \))
  \[ y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \]
  Substitute:
  \[ \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + k^2 \sum_{n=0}^{\infty} a_n x^n = 0 \]

Key: only satisfied if the net coefficient of each power of \( x \) is separately 0.
Consider the first couple of \( x^n \) terms:

\[ x^0 : \quad n = 2 \text{ in first, } n = 0 \text{ in second sum} \]

\[ 2 \cdot a_2 + k^2 a_0 = 0 \implies a_2 = -\frac{k^2 a_0}{2} \]

\[ x^4 : \quad n = 3 \text{ in first, } n = 1 \text{ in second sum} \]

\[ 3 \cdot 2 a_3 + k^2 a_1 = 0 \implies a_3 = -\frac{k^2 a_1}{3 \cdot 2} \]

Generalize (we have sufficient terms to do this in this case):

\[ x^{m-2} : \quad n = m \text{ in first, } n = m-2 \text{ in second} \]

\[ m(m-1)a_m + k^2 a_{m-2} = 0 \implies a_m = -\frac{k^2 a_{m-2}}{m(m-1)} \]

and repeat for \( a_{m+4} \) (and so on)

\[ a_m = \frac{k^2}{m(m-1)} \frac{-k^2 a_{m-4}}{(m-2)(m-3)} = \frac{(-k^2)^2 a_{m-4}}{m(m-1)(m-2)(m-3)} \left( \text{like factorial} \right) \]

Continue

\[ a_m = \begin{cases} \frac{k^m}{m^2} a_0 & \text{m even} \\ \frac{(-1)^m}{m} a_2 & \text{m odd} \end{cases} \]

\[ \implies y_2 = a_0 (x - \frac{kx^3}{2^1}) + \frac{kx^5}{4^1} + \ldots = a_0 \cos kx \quad \checkmark \]

\[ y_2 = a_1 (x - \frac{kx^3}{3^1}) + \frac{kx^5}{5^1} + \ldots = \frac{a_2}{k} \sin kx \]
Useful, but our examples seem to have singular points.⇒ Generalizations!

If isolated, use Laurent: \[ \sum_{n=m}^{\infty} a_n (x-x_0)^n \]
valid for \[ 0 < |x-x_0| < \rho \]

radius of convergence

If not isolated (e.g., branch point), then allow for non-integer values \[ y(x) = (x-x_0)^p \sum_{n=0}^{\infty} a_n (x-x_0)^n \]
⇒ Frobenius method

- power \( p \) of first non-vanishing term is a parameter to be determined.

See examples in Artken 9.5 and Lea 3.3.
- again, take derivatives (in special region) and substitute.
- \( p \) determined by finding coefficient of lowest power and setting it to zero. ⇒ indicial equation

Hypergeometric \[ (x^2-x) \frac{dy}{dx^2} + (2x-1) \frac{dy}{dx} + \frac{1}{4} y = 0 \]
singular at \( x=0,1 \). Look for solution about \( x=0 \).

Substitute (not true). Lowest coefficient is \( x^{p-1} \) with \[ -[p(p-1)+\frac{1}{2}p] a_0 = 0 \]
⇒ \( a_0 \neq 0 \) means \( p(p-\frac{1}{2}) = 0 \) ⇒ \( p = 0 \) or \( \frac{1}{2} \). Then match coefficients as before (next time).