834 Lecture 2

Before class:
- Have students sign sheet in order and take pictures.
  On board: "Please come have your picture taken."
  So I can learn names and faces."
- Log in to computer and set up with 834 page, hints,
  and Mathematica page. Start up Mathematica with some
  sample notebooks (including the one on 3D spherical integrals).
  Also load Jackson covers.
- Poll for Monday and Tuesday office hours

On the board:

\[ \mathbf{E} = \mathbf{S} \] warm-ups, plus "Spot the Error!" (answers in lecture notes)
\[ \nabla \cdot (\mathbf{V} \times \mathbf{W}) = \mathbf{W} \cdot (\nabla \times \mathbf{V}) - \mathbf{V} \cdot (\nabla \times \mathbf{W}) \] product rule
\[ \mathbf{V} \times (\mathbf{W} \times \mathbf{U}) = (\mathbf{V} \cdot \mathbf{W}) \mathbf{U} - (\mathbf{V} \cdot \mathbf{U}) \mathbf{W} + (\mathbf{W} \cdot \mathbf{U}) \mathbf{V} - (\mathbf{W} \cdot \mathbf{V}) \mathbf{U} \]
\[ \nabla \times (\phi \mathbf{V}) = 0 \]

\[ \mathbf{E} \times \mathbf{B} = \hat{z} \] (cylindrical)
  cylindrical: \( A_1 = 0, A_2 = 1, A_3 = 0 \) \[ \Rightarrow \hat{e}_z \mathbf{B} \] (sol.
  spherical: \( A_1 = 0, A_2 = 0, A_3 = 1 \) \[ \Rightarrow \] two terms!

"Spot the Error!" (what is wrong in each of these?)
\[ (\nabla \times \mathbf{V}) \cdot (\mathbf{V} \times \mathbf{W}) = E_{abc} \frac{\partial}{\partial x_b} V_c E_{def} \frac{\partial}{\partial x_e} W_f \]
\[ \nabla \times (\mathbf{V} \times \mathbf{W}) = E_{abc} \frac{\partial}{\partial x_b} E_{def} \frac{\partial}{\partial x_e} V_c \]

\[ (\mathbf{A} \times \mathbf{B})_d + (\mathbf{C} \times \mathbf{D})_d = E_{abc} A_b B_c + E_{def} C_e D_f \]
\[ E_{abc} \frac{\partial}{\partial x_b} E_{def} \frac{\partial}{\partial x_e} A_c \]
\[ [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_a = E_{abc} A_b E_{acd} B_c D_d \]
\[ (\mathbf{B} \times \mathbf{C}) \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \mathbf{C} \]

\[ \nabla \times (\mathbf{B} \cdot \mathbf{C}) = 0 \]

\[ \mathbf{B} \cdot \mathbf{C} \] is a scalar
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Answers to warm-ups:

\[
[\nabla \times (\nabla \times \mathbf{A})]_a = \varepsilon_{abc} \frac{\partial}{\partial x_b} (\nabla \times \mathbf{A})_c = \varepsilon_{abc} \frac{\partial}{\partial x_b} \varepsilon_{cde} V_d W_e
\]

or \( - \frac{1}{\rho} \) acts.

\[
= \varepsilon_{cde} \frac{\partial}{\partial x_b} V_b W_c = (\delta_{ad} \varepsilon_{cde} - \varepsilon_{ade} \delta_{bd}) \frac{\partial}{\partial x_b} V_d W_e
\]

\( \rho \) acts on both.

\[
\omega = \frac{\partial}{\partial x_b} V_b W_c - \frac{\partial}{\partial x_b} V_b W_c
\]

If \( a \nabla \) commute = \( \nabla \cdot (\nabla \times \mathbf{A}) = - (\nabla \times \nabla \omega) + (\nabla \cdot \mathbf{A}) \mathbf{V} - (\mathbf{V} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{V}
\]

\[

\nabla \cdot (\nabla \times \mathbf{A})_a = \frac{\partial}{\partial x_a} \varepsilon_{abc} V_b W_c = \varepsilon_{abc} \frac{\partial}{\partial x_a} V_b W_c
\]

\[
= \varepsilon_{abc} \left( \frac{\partial V_b}{\partial x_a} W_c + V_b \frac{\partial W_c}{\partial x_a} \right) \frac{\partial}{\partial x_a} V_b W_c
\]

If \( a \nabla \) commute = \( \varepsilon_{abc} W_c \frac{\partial}{\partial x_b} V_b + \varepsilon_{abc} V_b \frac{\partial}{\partial x_b} W_c
\]

\( \varepsilon_{bac} = - \varepsilon_{abc} \)

\[
= W_c (\nabla \times \mathbf{V})_b - V_b (\nabla \times \mathbf{A})_b = \mathbf{V} \cdot (\nabla \times \mathbf{A}) - (\nabla \times \mathbf{V}) \cdot \mathbf{A}
\]

\[
[\nabla \times (\phi \nabla \phi)]_a = \varepsilon_{abc} \frac{\partial}{\partial x_b} \left( \phi \frac{\partial}{\partial x_c} \phi \right) = \varepsilon_{abc} \left[ \frac{\partial^2 \phi}{\partial x_b \partial x_c} + \phi \frac{\partial^2}{\partial x_b \partial x_c} \phi \right]
\]

= 0 antisymmetric \times symmetric in both terms.
Follow-ups from Lecture 1:

- Recap on gradient $\nabla \Phi$ (what direction?):
  - Recall that a small step $\delta s$ changes $\Phi$ by $\delta \Phi(x_1, y_1, z_1) = \nabla \Phi \cdot \delta s$.
  - $\delta \Phi \cdot \delta s \approx \text{infinitesimal}$ just means small for physicists! So $(\delta r)^2 \text{small enough to neglect}.$
  - Intuitive association for direction of $\nabla \Phi$:
    - Recall that force $\mathbf{F} = -\nabla \Phi$ (potential) $\Rightarrow$ force is always toward lower potential ("downhill") $\Rightarrow \nabla \Phi$ points in direction of maximum positive change ("uphill").

- How do I know that $\nabla \Phi$ is actually a vector if $\Phi(x_1, y_1, z_1)$ is a scalar? (Arfken sec. 1.3):
  - Ans: Because it transforms like a vector from $x_i$ to $x'_j$ coordinate systems, scalar $\Phi(x) = \Phi(x_1) = \Phi(x_1, y_1, z_1) = \Phi(x'_1) = \Phi(x'_1, y'_1, z'_1)$.
  - Define $\alpha_{ij} = \frac{\partial x_i}{\partial x_j}$, e.g. $x' = x_1 \cos \alpha + y_1 \sin \alpha$$\Rightarrow \text{rotation by } \alpha \text{ about } z \text{ axis}$
    - $y' = -x_1 \sin \alpha + y_1 \cos \alpha$$\Rightarrow$ a vector transforms (i.e., $\Phi$ coordinates change) by
    - $V'_i = \alpha_{ij} V_j$ (summation convention!)

- Check $V = dx$:
  - $dx'_i = \alpha_{ij} dx_j = \frac{dx'_i}{dx_j} dx_j$ (really start with this!)
  - Inverse $\frac{dx'_i}{dx_j} = \alpha_{ji}$.
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Now the dot product of two vectors should be unchanged:

Scalar $\mathbf{A} \cdot \mathbf{B} = A_m B_n$ \Rightarrow $A_m = a_{lm} A_l, B_n = b_{ln} B_l$

\[ \Rightarrow \mathbf{A}' \cdot \mathbf{B}' = a_{lm} a_{ln} b_{lm} b_{ln} = a_{lm} A_l A_m B_n \text{ but equals } \mathbf{A} \cdot \mathbf{B} = A_m B_n \]

\[ \Rightarrow a_{lm} a_{ln} = S_{nm} \]

Best to think of matrix multiplication\footnote{transpose}:

\[ a_{mn} = (A^T)_{ml} \Rightarrow (A^T)_{ml} a_{ln} = S_{nm} \Rightarrow \text{orthogonal matrix} \]

\[ \Rightarrow A^T A = I \quad \text{identity matrix} \quad \text{or } A = A^{-1} \]

\[ \text{Then } \mathbf{A} \cdot \mathbf{B} = A_m (A^T)_{ml} A_l B_n = (a_\mathbf{a}^T) (\mathbf{a}) (\mathbf{b}) \]

\[ = a_m S_{mn} B_n = A_n B_n = \mathbf{A} \cdot \mathbf{B} \]

- a here is real \rightarrow different linear combinations of coordinates
  \footnote{unitary transformation}

- Helmholtz Theorem (Lea 1.4)
  Any vector $\mathbf{F}$ can be expressed as the sum of a gradient of a scalar and the curl of a vector:

\[ \mathbf{F} = \nabla \phi + \nabla \times \mathbf{A} \]

\[ \nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \phi + \nabla \times \mathbf{A}) = \nabla^2 \phi \quad \text{only } \phi \text{ contributes to divergence} \]

\[ \nabla \times \mathbf{F} = \nabla \times (\nabla \phi + \nabla \times \mathbf{A}) \quad \text{only } \mathbf{A} \text{ contributes to curl} \]
Continuations from Lecture 1 notes:

- pg.10 Other formulas from Jackson: covers central vectors
- pg.12 Vector calculus Theorem prototype: simplest example
- pg.14 Partial integration in vector calculus

pg.13 + PS#1: Applying vector calculus Theorems
  - quickly run through examples 1.2 and 1.3 from Lea Chap.1
  - Question: What dictates the best choice of coordinates
to evaluate a volume or surface integral?
  - Is it the integrand or the limits?
  - Not always a fixed answer or an optimal answer.
  - Often it is the limits \( \Rightarrow \) sets the geometry.
  - Comments on homework PS#1.5 and 1.6.

Prob. 1.5

\[
\begin{align*}
\begin{array}{c}
3 \\
4
\end{array}
\end{align*}
\rightarrow_x
\begin{align*}
\begin{array}{c}
\hfill 5 \\
\hfill 5
\end{array}
\end{align*}
\]

Nature of C here suggests Cartesian
is by far the easiest.

How does direction of traversing C come in?
- What is \( \hat{\mathbf{n}} \)? What if you reversed the direction?
- Recall curl intuition: put small paddle wheel in moving fluid, \( \hat{\mathbf{B}} \times \hat{\mathbf{V}} \) aligned with axis tells how much it will rotate.

Prob. 1.6 Integration over a hemisphere \( \Rightarrow \) initial and \( \Phi \times \mathbf{V} \leq 0 \).
  - Which coordinates?
  - Spherical coordinate integration with Mathematica
    \( \Rightarrow \) go through sample notebook on Mathematica webpage

\( \hat{\mathbf{n}} \) is positive normal to surface:
- if closed surface, \( \hat{\mathbf{n}} \) outward
- if open, right-hand rule on direction of boundary contours determines \( \hat{\mathbf{n}} \).
Complex Analysis

Introductory remarks:

- Complex analysis is a big subject, generally a semester course by itself in a math department.
- We'll spend about 2½ periods.
- So we have to focus our attention and concentrate on results, not proofs.

Partial list of motivations for complex analysis (see Arfken)
- Real physical quantities can become complex as physical theory is made more general.
- Add absorption and real index of refraction of light becomes complex.
- Allow decay and real energies become complex.
- $k ightarrow i\omega$ relates Helmholtz to diffusion equation.
- $f ightarrow \tau$ imaginary time equations ("Euclidean") are key to numerical solutions of field theories (and other Monte Carlo methods like Green's Function Monte Carlo).
- Insight into and tools for solving differential equations.
- Integrals in complex plane have many applications.

Core competencies

1) Comfortable with manipulations of complex variables and functions.
   - Both Cartesian and polar form
2) Evaluation of integrals. Requires:
   - Understanding of analyticity (Cauchy-Riemann relations).
   - Singularities (poles, branch points, essential singularities).
   - Laurent series expansions.
   - Specific applications of contour integration (residues, etc).
3) Dispersion relations (if time).

Omitted: conformal mapping, proofs of any of this!
Reminders & complex number representation...
must know

\[ z = x + iy \iff (x, y) \]

\[
\begin{align*}
& i^2 = -1, \\
& i^4 = 1
\end{align*}
\]

"Argand diagram" or "complex Z plane"

In xy, \[ |z| \] \text{ is notation}

\[ \text{usually choose } 0 \leq \theta < 2\pi \]

or \[ -\pi < \theta \leq 0 \] to get \[ \theta \] unique

\[ \text{but clearly some freedom!} \]

\[ 3e^{i\pi/3} = 3\left(\cos\left(\frac{\pi}{3} + 2\pi n\right) + i\sin\left(\frac{\pi}{3} + 2\pi n\right)\right) = 3e^{i\pi/3} \]

Complex conjugate \[ z^* = x - iy \]

\[ zz^* = x^2 + y^2 \geq 0 \]

Recall how simple manipulations work through examples.

Let's problem 2.2: Find an expression for \[ \cos 3\theta \] and for \[ \sin 3\theta \] in terms of \[ \cos \theta \] and \[ \sin \theta \]

Connection with complex numbers \[ e^i\theta = \cos \theta + i\sin \theta \]

How do I know this? Taylor series probably best?

\[ e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + \sum_{n=\text{even}} \frac{(i\theta)^n}{n!} \]

\[
\begin{align*}
& = 1 - \theta^2 + \frac{\theta^4}{4!} - \cdots \quad \text{cos \( \theta \)} \\
& + \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \quad \text{sin \( \theta \)}
\end{align*}
\]

Question: is \[ e^{i\theta^2} = \cos \theta + i\sin \theta \] for complex \( \theta \)?

(ans: yes, Taylor series expansion holds)

\[
\begin{align*}
& \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} , \\
& \cosh(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} , \\
& \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} , \\
& \sinh(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2}
\end{align*}
\]
Try Cos[6θ] or Cos[6θ] = Cos[θ]^6 - 3Cos[θ]^4 + 3Cos[θ]^2 - 1

For example, can Mathematica tell us this?

Try these on computer

Cos[6θ] = -2*Cos[θ]^6 + 3Cos[θ]^4 - 3Cos[θ]^2 + 1

How do we know? Let all terms on one side:

cos[complex (real part)] = -2 real part

so each separately two

Thus, we know, real and imaginary parts must be separately equal.

z^3 = \cos(θ) + i\sin(θ)

\cos^3θ + 3\cos(θ)^2(\sin(θ)) + 3\cos(θ)(\sin(θ))^2 + \sin^3θ

z = \cos(θ)

\sin(θ)
9.26/1

**Functions of a Complex Variable**

\[ z = x + iy \Rightarrow w = f(z) = u(z) + iv(z) \]

- real valued functions of complex argument

Ordinary function \( f(x) \) is a mapping from \( x \in \mathbb{R} \) to \( f(x) \in \mathbb{R} \)

- "element of" real numbers

\[ \Rightarrow \text{represent as graph} \]

Complex \( z = x + iy \) to \( w = u + iv \)

is mapping from complex \( z \) plane to complex \( w \) plane

Hard to draw an analog to \( f(x) \) vs. \( x \) (need 4-dimensional representation)

- Instead, look at a line or a region maps. (More later)

Strange behavior can happen. Consider \( w = z^{1/2} \)

\[ w = z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2} \]

- well defined positive square root

\[ \sqrt{1} = e^{i\pi/2} \Rightarrow 1^{1/2} e^{i\pi/2} = +i \]

\[ \Rightarrow 1^{1/2} e^{-i\pi/2} = -i \]

- different answers depending on how \( z \) is specified.

- Looks like same point in \( Z \) plane maps to two different values.

- Introduce a branch cut to define \( z^{1/2} \) as single valued.

- Conventional:

  - \( \mathbb{C} \) to put on negative real axis

  - Branch point for plane so bottom and top not connected.

  - More later: for now we've simply cut
Other functions with this behavior: \( z^{1/n}, \ln z \)

\[
\begin{align*}
  w &= \ln z = \ln (re^{i\theta}) = (\ln r) + i\theta, \\
  \text{so } &0 \leq r < \infty, 0 \leq \theta < \pi, \\
  \Rightarrow \text{branch point at } z=0, \text{ multiple sheets}
\end{align*}
\]

Consider \( z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n} \) is mapped to \( 0 \leq \theta < 2\pi \)

\Rightarrow multiple copies

Suppose \( n=4 \) \( \Rightarrow w = z^{1/4} \) or \( z = w^4 = 2 \). What are solutions?

\[
\begin{align*}
  z: &\quad 2e^{\pi i/4}, 2e^{3\pi i/4}, 2e^{5\pi i/4}, 2e^{7\pi i/4} \\
  w: &\quad 2^{1/4}, 2^{1/4}e^{i\pi/4}, 2^{1/4}e^{3\pi/4}, 2^{1/4}e^{5\pi/4}
\end{align*}
\]

\Rightarrow four solutions

\[
\begin{array}{cccc}
  \downarrow & \downarrow & \downarrow & \downarrow \\
  2 & 1 & -1 & -2 \\
\end{array}
\]

Solutions are vertices of square

\Rightarrow 4 solutions to \( w^4 = 2 \).

for \( z^{1/n} \), \( n \)-polygon starting on x-axis
Let's put aside the tricky functions and think of nice ones: continuous and smooth.

Smooth. For ordinary functions means derivatives exist from both directions:
\[
\frac{df}{dx} = \lim_{x \to 0} \frac{f(x+dx) - f(x)}{dx} \quad \text{smooth} \quad \text{from both directions!}
\]

For complex functions, derivative defined similarly:
\[
\frac{df}{dz} = \lim_{z \to 0} \frac{f(z+dz) - f(z)}{dz} \quad \text{but } z+dz \text{ can approach } z \text{ from many directions!}
\]

requiring the same answer is a powerful constraint \( \Rightarrow \) analytic functions

Sufficient to consider just y-axis as x-axis approach.

Do it:
\[
\frac{df}{dz} = \lim_{dz \to 0} \frac{f(x+dx,y)+i\cdot f(x,y+dy)-[f(x,y)+i\cdot f(x,y)]}{dz} = \frac{du}{dx} + i \frac{dv}{dx}
\]
\[
\frac{dy}{dz} = \lim_{dz \to 0} \frac{f(x,y)+i\cdot f(x,y)-[f(x,y)+i\cdot f(x,y)]}{dz} = \frac{1}{i} \left[ \frac{du}{dy} + \frac{dv}{dy} \right]
\]

equate real:
\[
\frac{du}{dx} = \frac{dv}{dy}
\]
equate imaginary:
\[
\frac{dv}{dx} = -\frac{du}{dy}
\]

Cauchy-Riemann equations \( \Rightarrow \) powerful constraint!
If differentiable at $z_0$ and nearby, then analytic at $z = z_0$.

If true for all $z$, then function is entire.

Check $f(z) = z^3$ for analyticity. Are C-R equations satisfied?

$$f(z) = z^3 = (x + iy)^3 = x^3 + 3x^2iy + 3xiy^2 + (iy)^3$$

$$= x^3 - 3x^2y + i(3xy - y^3)$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$
$$\frac{\partial v}{\partial y} = 6xy$$

$$\frac{\partial u}{\partial y} = 0$$
$$\frac{\partial v}{\partial x} = 0$$

$\Rightarrow$ C-R satisfied and partial derivatives continuous everywhere $\Rightarrow$ analytic

Another type of problem:

Find analytic function $w(z) = u(x, y) + iv(x, y)$

given that $u(x, y) = x^3 - 3xy^2$ [so find $v(x, y)$]

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \Rightarrow v = 3x^2y - y^3 + f(x) \leq \text{any function of x}$

$$-\frac{\partial u}{\partial y} = 6xy$$

$$\frac{\partial v}{\partial x} = 6xy + \frac{df}{dx} \Rightarrow \text{equal if } \frac{df}{dx} = 0 \text{ or } f = \text{const.}$

$\Rightarrow w(z) = x^3 - 3xy^2 + i(3x^2y - y^3 + \text{const.})$ is analytic!

Is $\text{Re} z = x$ analytic?

$u = x, v = 0$

$\Rightarrow \frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial y} = 0$ not equal! So no.
Try \( f(z) = \frac{1}{z} \) for satisfying C-R equations.

\[
\frac{\partial u}{\partial x} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x^2+y^2} \Rightarrow u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}
\]

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{x^2+y^2} + \frac{x}{(x^2+y^2)^2} \cdot 2x = \frac{1}{(x^2+y^2)^2} (x^2+y^2-2x^2) = \frac{y^2-x^2}{(x^2+y^2)^2}
\]

\[
\frac{\partial^2 v}{\partial y^2} = -\frac{1}{x^2+y^2} \cdot (-y) \cdot \frac{4}{(x^2+y^2)^2} \cdot 2y = \frac{1}{(x^2+y^2)^2} (2y^2-x^2) = \frac{y^2-x^2}{(x^2+y^2)^2}
\]

\[
\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}
\]

Similarly, \( \frac{\partial u}{\partial y} = -\frac{x}{x^2+y^2} \cdot 2y \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow u(x,y) = f(z)
\]

So analytic except when \( x = y = 0 \Rightarrow a \equiv z = 0 \)

\( f(z=0) = \infty \), called a "simple pole" (more later).

But now try \( f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x+iy}{x^2+y^2} \Rightarrow u = \frac{x}{x^2+y^2}, \quad v = \frac{y}{x^2+y^2}
\]

\[
\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} \neq \frac{\partial u}{\partial x} \Rightarrow \text{not analytic. Close doesn't count!}
\]

Can't just assign \( u \) and \( v \) functions and get analytic \( w \).

Next: Integrals of \( f(z) \)!