

## 7. 780.20 Session 7

### a. Follow-ups to Session 6

- **How do we implement 4th-order Runge-Kutta for a system of  $N$  coupled, linear differential equations?** The formulas from Landau/Paez are

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4), \quad (7.1)$$

with

$$\begin{aligned} \mathbf{k}_1 &= h\mathbf{f}(\mathbf{y}_i, t_i); & \mathbf{k}_2 &= h\mathbf{f}(\underbrace{\mathbf{y}_i + \mathbf{k}_1/2}_{\alpha_1}, t_i + h/2); \\ \mathbf{k}_3 &= h\mathbf{f}(\underbrace{\mathbf{y}_i + \mathbf{k}_2/2}_{\alpha_2}, t_i + h/2); & \mathbf{k}_4 &= h\mathbf{f}(\underbrace{\mathbf{y}_i + \mathbf{k}_3}_{\alpha_3}, t_i + h). \end{aligned} \quad (7.2)$$

The procedure for a function to take us from the vector  $\mathbf{y}_i$  of length  $N$  (which is input) to  $\mathbf{y}_i$  (which is returned), is

1. calculate  $\mathbf{k}_1$  and  $\alpha_1$ ;
2. calculate  $\mathbf{k}_2$  and  $\alpha_2$ ;
3. calculate  $\mathbf{k}_3$  and  $\alpha_3$ ;
4. calculate  $\mathbf{k}_4$ ;
5. calculate  $\mathbf{y}_{i+1}$ ,

which therefore requires five loops (since we have to calculate the  $N$  elements of each vector at every step). This is what is implemented in the `runge4` function (except that  $\alpha_{1,2,3}$  are called  $y_{1,2,3}$ ).

- **What  $h$  should we use for 4th-order Runge-Kutta if we don't know the exact answer?** [This discussion is based on Numerical Recipes, Sect. 16.2 [2].] One way is to compare stepping from  $y(t)$  to  $y(t + 2h)$  in one step (call the answer  $y_1$ ) and in two steps (call the answer  $y_2$ ). From the handout or Numerical Recipes, we will find that the error for Runge-Kutta with a step  $h$  goes like  $h^5\phi$ , where  $\phi \approx y^{(5)}(t)/5!$  (which comes from a Taylor expansion). So  $y_1$  and  $y_2$  should be the same up to this error, except that there is one step by  $2h$  in one case and two steps by  $h$  in the other:

$$\text{one step} \quad \implies \quad y(t + 2h) \approx y_1 + (2h)^5\phi + \mathcal{O}(h^6), \quad (7.3)$$

$$\text{two steps} \quad \implies \quad y(t + 2h) \approx y_2 + 2(h)^5\phi + \mathcal{O}(h^6). \quad (7.4)$$

Then  $\Delta \equiv y_2 - y_1$  is a measure of the truncation error. [Note that we could use Richardson extrapolation here; do you see how?] If the desired accuracy is  $\Delta_0 = \epsilon y$  (that is,  $\epsilon$  is the desired *relative* error), then having used an arbitrary  $h_1$  to get  $\Delta_1$  (that is, we make a first guess), we can determine the  $h_0$  needed to get  $\Delta_0$  from

$$\frac{\Delta_0}{\Delta_1} = \left(\frac{h_0}{h_1}\right)^5 \quad \implies \quad h_0 = \left|\frac{\Delta_0}{\Delta_1}\right|^{1/5} h_1. \quad (7.5)$$

We could make this adjustment at each step, and so make the solution algorithm *adaptive*. GSL has such an adaptive routine using 4th-order Runge-Kutta.

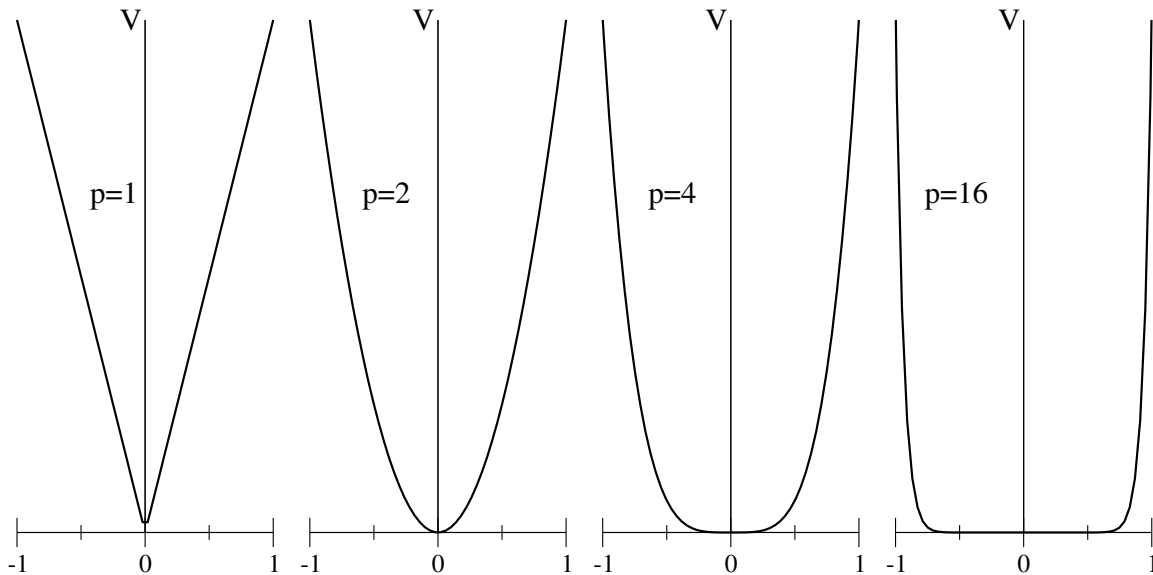
## b. Forced, Nonlinear Oscillator

In Session 7, we'll apply our differential equation solving routines to a basic problem: a forced (that is, driven), oscillator. We'll give an overview here; see also the Landau/Paez handouts [1].

The force we'll use is a generalization of Hooke's law for springs:

$$F_k(x) = \begin{cases} -kx^{p-1}, & x > 0, \\ +k|x|^{p-1}, & x < 0. \end{cases} \quad (7.6)$$

The parameter  $p$  is defined so the potential energy is  $V(x) = \frac{1}{p}k|x|^p$ . So  $p = 2$  reduces to the usual spring. Here is the potential energy as a function of  $x$  for a range of  $p$ :



Notice that an infinite square-well potential is found in the  $p \rightarrow \infty$  limit. (Classically, the mass bounces back and forth between  $x = \pm 1$ , moving freely in between bounces.)

We'll add to this an external, time-dependent force  $F_{\text{ext}}(x, t)$ . Newton's second law gives us a differential equation for  $x(t)$ :

$$F = Ma \implies \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{1}{M}[F_k(x) + F_{\text{ext}}(x, t)]; \quad v \equiv \frac{dx}{dt}. \quad (7.7)$$

To use the Runge-Kutta routine in `diffeq_routines.cpp`, we define the  $y^{(i)}$ 's as

$$y^{(0)}(t) = x(t), \quad y^{(1)}(t) = v(t), \quad (7.8)$$

so that the coupled equations are

$$\frac{dy^{(0)}}{dt} = y^{(1)}, \quad \frac{dy^{(1)}}{dt} = \frac{1}{M}F_k(x) + \frac{1}{M}F_{\text{ext}}(x, t) \quad (7.9)$$

(with the “rhs” functions defined by the right sides of these equations). Since we have two equations, we need two initial conditions, which are the initial position  $x_0$  and initial velocity  $v_0$ :

$$y^{(0)}(t=0) = x_0, \quad y^{(1)}(t=0) = v_0. \quad (7.10)$$

Check the `diffeq_oscillations.cpp` code to see how this works in practice.

Some things to note:

- “Anharmonic” oscillator means  $p \neq 2$ . What is special about a harmonic oscillator? (E.g., what is the relation between the amplitude and the period when  $p = 2$ ? Is this true for  $p \neq 2$ ?)
- The total energy is the sum of the kinetic energy (KE) and potential energy (PE):

$$E(t) = \text{KE}(t) + \text{PE}(t) = \frac{1}{2}Mv(t)^2 + V[x(t)]. \quad (7.11)$$

What do you expect the plot of this to be like if undriven? If driven with a sinusoidal external force? What if the oscillator is damped?

- In the undamped, undriven case, the total energy is conserved, the the kinetic and potential energies are time dependent. However, when they are averaged over a period, they satisfy a *virial theorem*:

$$\langle \text{KE} \rangle = \frac{p}{2} \langle \text{PE} \rangle, \quad (7.12)$$

where the  $\langle \rangle$ 's denote a time average. Do you know how to derive this? [Hint: Consider the time average over a period of the quantity  $Mvx$ .] How would you check numerically if the virial theorem is satisfied?

### c. Damped Oscillations

In the real world of macroscopic physical systems, there is friction, which will damp oscillations. Landau and Paez present three models for friction: *static*, *kinetic* (or sliding), and *viscous*, with the associated force laws [1]:

$$F_f \leq -\mu_s N, \quad \text{static}, \quad (7.13)$$

$$F_f = -\mu_k N \frac{v}{|v|}, \quad \text{kinetic}, \quad (7.14)$$

$$F_f = -bv, \quad \text{viscous}, \quad (7.15)$$

where  $N$  is the normal force, the  $\mu$ 's are coefficients of static and kinetic friction,  $b$  is a damping parameter, and  $v$  is the velocity. Note that the  $v/|v|$  piece of the kinetic friction is just  $\pm 1$ , which ensures that the friction opposes the motion.

There are other types of viscous friction (e.g., other powers of  $v$  can appear) under different circumstances, but we'll use this one in Session 7. When included in a harmonic or slightly anharmonic oscillator, we can identify three regimes (look for them!) based on how the damping parameter to mass ratio  $b/M$  compares to the natural (undamped, undriven) frequency  $\omega_0$ :

1. **Underdamped:** The solution oscillates within bounds that decay exponentially (i.e., there is an *envelope* when  $\frac{b}{2M} < \omega_0$ ).
2. **Critically damped:** The solution goes directly to the equilibrium position without any oscillations when  $\frac{b}{2M} = \omega_0$ .
3. **Overdamped:** The solution decays slowly, reaching the equilibrium position only after an infinite time (in principle), when  $\frac{b}{2M} > \omega_0$ .

Can you think of applications to real life where you would like the oscillations to be critically damped (or, really, just slight underdamped)?

#### d. Resonance

If we have a sinusoidal, external driving force

$$F_{\text{ext}}(t) = F_0 \sin \omega t , \quad (7.16)$$

then we can observe *resonance* for either harmonic or anharmonic oscillations. (We can also see beats, as described in Landau/Paez 11.9.) When a *harmonic* oscillator ( $p = 2$ ) that has natural frequency  $\omega_0$  is driven at  $\omega = \omega_0$ , the system resonates. If there is no damping, the amplitude of the oscillation increases with time indefinitely (well, until the approximation of harmonic forces breaks down). How does the nature of the resonance change when  $p \neq 2$ ?

#### e. Phase-Space Plots

The classical solution to the one-dimensional oscillator is conventionally given as the position  $x(t)$  and the velocity  $v(t)$  as functions of time. A useful visualization of the motion is obtained by plotting  $v(t)$  versus  $x(t)$ , which is called a *phase-space plot*. Here are some questions you can ask using phase-space plots of the oscillator studied in this session:

- If we followed points being plotted as time increases, is the motion clockwise or counter-clockwise? Is this a general result?
- What is the shape of the closed curves for a harmonic  $p = 2$  oscillator?
- What differences are there in the shape of the orbits for anharmonic oscillations? Do the initial conditions matter?

#### f. References

- [1] R.H. Landau and M.J. Paez, *Computational Physics: Problem Solving with Computers* (Wiley-Interscience, 1997).
- [2] W. Press *et al.*, *Numerical Recipes in C* (Cambridge, 1992). Individual chapters are available online from <http://lib-www.lanl.gov/numerical/bookcpdf.html>. There are also versions for Fortran and C++.