Physics 263: Example of Using the Cofactor Matrix

Let's do problem 8.3.5: *For the matrix*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

find the cofactor matrix and the inverse. Verify that your inverse does the job.

Definition 8.5 says: The **cofactor matrix** M_C has elements

$$(M_C)_{ij} = (-1)^{i+j} \times (\text{the determinant of the matrix with row } i \text{ and column } j \text{ deleted}) \ .$$

So in this case:

$$M_C = \begin{bmatrix} +\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} & +\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 8 & 10 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 7 & 10 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} +2 & +2 & -3 \\ +4 & -11 & +6 \\ -3 & +6 & -3 \end{bmatrix}$$

According to Definition 8.6, The determinant of a 3×3 matrix M is

$$|M| = M_{11}(M_C)_{11} + M_{12}(M_C)_{12} + M_{13}(M_C)_{13} \ .$$

This is an example of *expansion in minors*. In the present case, we find

$$|M| = (+1)\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} - (+2)\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} + (+3)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 1 \cdot 2 + (-2)(-2) + 3(-3) = -3 \ .$$

If we use the second row instead, we get the same result:

$$|M| = -(+4)\begin{vmatrix} 2 & 3 \\ 8 & 10 \end{vmatrix} + (+5)\begin{vmatrix} 1 & 3 \\ 7 & 10 \end{vmatrix} - (+6)\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = (-4)(-4) + 5(-11) + (-6)(-6) = -3 \ .$$

The inverse of M is then given by

$$M^{-1} = \frac{M_C^T}{|M|} = -\frac{1}{3}\begin{bmatrix} +2 & +4 & -3 \\ +2 & -11 & +6 \\ -3 & +6 & -3 \end{bmatrix} = \begin{bmatrix} -2/3 & -4/3 & +1 \\ -2/3 & 11/3 & -2 \\ +1 & -2 & +1 \end{bmatrix}$$

We can verify our result by checking that MM^{-1} is the three-by-three identity matrix.