Name: ______________________________

Physics 263: Chapter 7 Practice Problems, Part I

These are practice for the first part of Chapter 7 of BTM, which covers vectors and gradients.

Vector Review: Dot, Cross, and Triple Products

We use the standard notation for cartesian unit vectors $\widehat{\mathbf{i}} \equiv \widehat{\mathbf{x}}$, $\widehat{\mathbf{j}} \equiv \widehat{\mathbf{y}}$, $\widehat{\mathbf{k}} \equiv \widehat{\mathbf{z}}$, which have the following properties under the dot product:

$$\widehat{\mathbf{i}} \cdot \widehat{\mathbf{i}} = \widehat{\mathbf{j}} \cdot \widehat{\mathbf{j}} = \widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}} = 1 \ , \qquad \widehat{\mathbf{i}} \cdot \widehat{\mathbf{j}} = \widehat{\mathbf{j}} \cdot \widehat{\mathbf{k}} = \widehat{\mathbf{k}} \cdot \widehat{\mathbf{i}} = 0 \ ,$$

and the following properties under the cross product:

$$\widehat{\mathbf{i}} \times \widehat{\mathbf{i}} = \widehat{\mathbf{j}} \times \widehat{\mathbf{j}} = \widehat{\mathbf{k}} \times \widehat{\mathbf{k}} = 0 \ , \qquad \widehat{\mathbf{i}} \times \widehat{\mathbf{j}} = -\widehat{\mathbf{j}} \times \widehat{\mathbf{i}} = \widehat{\mathbf{k}} \ , \quad \widehat{\mathbf{j}} \times \widehat{\mathbf{k}} = -\widehat{\mathbf{k}} \times \widehat{\mathbf{j}} = \widehat{\mathbf{i}} \ , \quad \widehat{\mathbf{k}} \times \widehat{\mathbf{i}} = -\widehat{\mathbf{i}} \times \widehat{\mathbf{k}} = \widehat{\mathbf{j}} \ .$$

Two vectors are *orthogonal* (at right angles) if their dot product is zero.

For the following vectors,

$$\overrightarrow{\mathbf{V}} = 2\widehat{\mathbf{i}} + 3\widehat{\mathbf{j}} - 4\widehat{\mathbf{k}} \ , \qquad \overrightarrow{\mathbf{W}} = \widehat{\mathbf{i}} - 2\widehat{\mathbf{j}} - \widehat{\mathbf{k}} \ , \qquad \overrightarrow{\mathbf{Z}} = -2\widehat{\mathbf{i}} - 3\widehat{\mathbf{j}} + 4\widehat{\mathbf{k}} \ .$$

1. Check which pairs of vectors are orthogonal.

2. Find the cross products $\overrightarrow{\mathbf{V}} \times \overrightarrow{\mathbf{W}}$, $\overrightarrow{\mathbf{W}} \times \overrightarrow{\mathbf{Z}}$, and $\overrightarrow{\mathbf{Z}} \times \overrightarrow{\mathbf{V}}$, then find the magnitudes $|\overrightarrow{\mathbf{V}} \times \overrightarrow{\mathbf{W}}|$, $|\overrightarrow{\mathbf{W}} \times \overrightarrow{\mathbf{Z}}|$, and $|\overrightarrow{\mathbf{Z}} \times \overrightarrow{\mathbf{V}}|$.

3. Find the triple products $\overrightarrow{\mathbf{V}} \cdot (\overrightarrow{\mathbf{W}} \times \overrightarrow{\mathbf{Z}})$ and $\overrightarrow{\mathbf{W}} \cdot (\overrightarrow{\mathbf{V}} \times \overrightarrow{\mathbf{Z}})$.

Simple Line Integrals and Surface Integrals

1. Calculate the line integral of $\overrightarrow{\mathbf{F}} = (x^2 - y)\,\widehat{\mathbf{i}} + (y^2 + x)\,\widehat{\mathbf{j}}$ between the points $(0, 1)$ and $(1, 2)$ along a straight line.

2. Calculate the line integral of $\vec{\mathbf{F}} = (x^2 - y)\,\hat{\mathbf{i}} + (y^2 + x)\,\hat{\mathbf{j}}$ between the points $(0, 1)$ and $(1, 2)$ along the curve $x = t$, $y = t^2 + 1$.

3. Calculate the line integral of $\vec{\mathbf{F}} = (3x^2 - 6yz)\,\hat{\mathbf{i}} + (2y + 3xz)\,\hat{\mathbf{j}} + (1 - 4xyz^2)\,\hat{\mathbf{k}}$ between the points $(0, 0, 0)$ and $(1, 1, 1)$ along a straight line and along the curve $x = t$, $y = t^2$, $z = t^3$.

4. Find the surface integral of $\vec{\mathbf{V}} = (2x - z)\,\hat{\mathbf{i}} + x^2 y\,\hat{\mathbf{j}} + -xz^2\,\hat{\mathbf{k}}$ over a unit cube in the first octant, with one corner at the origin.

Gradients

1. If $\phi = 2x^2 y - xz^3$, find $\boldsymbol{\nabla}\phi$.

2. The temperature on your favorite hot volcanic mountain is given by $T(x, y) = 3xy^2 + x^4 y$. *(i)* If you are located at $(x = 1, y = 1)$, in which direction should you walk to best escape the heat? *(ii)* If you steps are $1/10$ units long, by how much will the temperature drop after the first step? (Work to first order.)