

Basics of Complex Numbers (I)

1. General

- $i \equiv \sqrt{-1}$, so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and then it starts over again ($i^5 = i$, $i^6 = -1, \dots$).
- Any complex number z can be written as the sum of a real part and an imaginary part:

$$z = [\operatorname{Re} z] + i[\operatorname{Im} z] ,$$

where the numbers or variables in the $[\]$'s are *real*. So $z = x + yi$ with x and y real is in this form but $w = 1/(a + bi)$ is *not* (see "Rationalizing" below). Thus, $\operatorname{Im} z = y$, but $\operatorname{Re} w \neq 1/a$.

- **Complex Conjugate:** The complex conjugate of z , which is written as z^* , is found by changing the sign of every i in z :

$$z^* = [\operatorname{Re} z] - i[\operatorname{Im} z] \quad \text{so if } z = \frac{1}{a + bi} , \text{ then } z^* = \frac{1}{a - bi} .$$

Note: There may be "hidden" i 's in the variables; if a is a complex number, then $z^* = 1/(a^* - bi)$.

- **Magnitude:** The magnitude squared of a complex number z is:

$$zz^* \equiv |z|^2 = [\operatorname{Re} z]^2 - (i)^2[\operatorname{Im} z]^2 = [\operatorname{Re} z]^2 + [\operatorname{Im} z]^2 \geq 0 ,$$

where the last equality shows that the magnitude is positive (except when $z = 0$).

Basic rule: if you need to make something real, multiply by its complex conjugate.

2. **Rationalizing:** We can apply this rule to "rationalize" a complex number such as $z = 1/(a + bi)$. Make the denominator real by multiplying by one in the form of the complex conjugate on top and bottom:

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2}$$

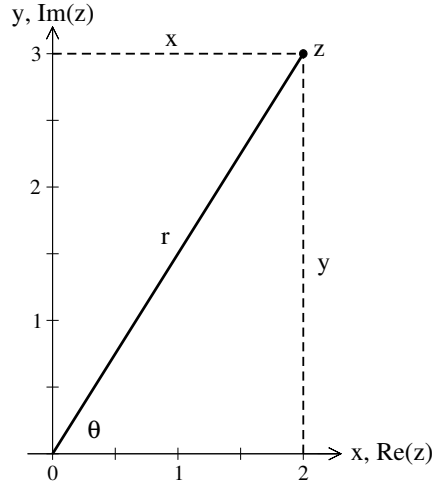
so $\operatorname{Re} z = a/(a^2 + b^2)$ and $\operatorname{Im} z = -b/(a^2 + b^2)$.

3. The Complex x-y Plane

- **Rectangular form:** Any complex number z can be uniquely represented as a point in the x - y plane, where the x -coordinate is $\operatorname{Re} z$ and the y -coordinate is $\operatorname{Im} z$.
 - You can think of i as a unit vector in the "imaginary" (y) direction.
 - The magnitude of z is just the length of the vector from the origin.
- **Polar form:** We can also write z in polar form as (see figure):

$$z = r e^{i\theta} = r \cos \theta + i r \sin \theta ,$$

where r and θ are real and equal to the length and angle of the vector.



- The complex conjugate of $z = r e^{i\theta}$ is $z^* = r e^{-i\theta}$.
- Thus the magnitude is $|z| = \sqrt{z z^*} = r$.
- Rationalizing:

$$\frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta} .$$

4. Multiplying Complex Numbers

- Multiplication is distributive: $(a + b i) \times (c + d i) = (ac - bd) + i(ad + bc)$.
- In polar form, we multiply the r 's and add the θ 's: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.

5. Euler's Theorem and other Goodies:

- From the polar form, we have:

$$e^{i\theta} = \cos \theta + i \sin \theta .$$

- Special values:

$$e^{2\pi i} = 1 \quad e^{i\pi} = -1 \quad e^{i\pi/2} = i \quad e^{i\pi/4} = 1/\sqrt{2} + i/\sqrt{2}$$

- We can rewrite sin and cos:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} ,$$

which can be very useful, since it is generally easier to work with exponentials than trigonometric functions.

- DeMoivre's Theorem:

$$z^n = (r e^{i\theta})^n = r^n e^{in\theta} = r^n [\cos(n\theta) + i \sin(n\theta)] .$$

We can also write the theorem in the form:

$$z^{1/n} = r^{1/n} [\cos(\theta/n) + i \sin(\theta/n)] ,$$

which is great for taking the square root, cube root, etc. of complex numbers!