

Name: \_\_\_\_\_

## Physics 263: 1094 Session 10

**Directions:** Step through this sheet and write down answers to the questions/problems in italics as you go. Hand it in at the end of the period for credit. Please work in pairs (but hand in individual sheets) and feel free to check your answers against those from other groups.

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### I. Orthonormal Bases (15 minutes)

1. **Expansion in an orthonormal basis.** We'll do the first part of BTM problem (9.2.1) to illustrate expanding in an orthonormal basis. The basis vectors are

$$|I\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |II\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Define these in MATLAB (e.g., call them `Iket` and `IIket`) and calculate (see the cheat-sheet)  $\langle I|I\rangle$ ,  $\langle II|II\rangle$ , and  $\langle I|II\rangle$ , remembering that  $\langle I| = (|I\rangle)^\dagger$  or `Ibra = Iket'`. Is this an orthonormal basis? Explain.

Now we look for  $v_I$  and  $v_{II}$ , where  $|V\rangle = v_I|I\rangle + v_{II}|II\rangle$ . Determine  $v_I$  and  $v_{II}$  using  $v_I = \langle I|V\rangle$  and  $v_{II} = \langle II|V\rangle$  for (answers in the back of the book)

$$V = \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix}$$

2. **Gram-Schmidt orthogonalization.** Let's carry out a variation of BTM problem (9.2.2), by forming an orthonormal basis in two dimensions starting with

$$|A\rangle = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad |B\rangle = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Carry out these steps (giving the answers to each part):

- Find  $|1\rangle$  (call it  $\mathbf{v1}$ ), which is the unit vector in the direction of  $|A\rangle$  (see the cheat sheet).
- Find  $|2'\rangle$  (call it  $\mathbf{v2p}$ ) from

$$|2'\rangle = |B\rangle - |1\rangle\langle 1|B\rangle.$$

- Finally, find  $|2\rangle$  (call it  $\mathbf{v2}$ ), which is the unit vector in the direction of  $|2'\rangle$ .
- Verify that the basis is orthonormal. *How did you verify it?*

## II. Eigenvalues and Eigenvectors (25 minutes)

If  $A$  is a  $N \times N$  square matrix, then an *eigenvector*  $V_i$  of  $A$  is a column vector with the property that  $AV_i$  is just a multiple of  $V_i$  (rather than an unrelated vector), which means

$$AV_i = \lambda_i V_i \quad \text{or} \quad (A - \lambda I)V = 0 ,$$

where  $\lambda_i$  is a real or complex number called the *eigenvalue*. We expect  $N$  eigenvalues and corresponding eigenvectors. We can find the eigenvalues by solving for the solutions to the *characteristic polynomial* in  $\lambda$ , which is the  $n^{\text{th}}$  degree polynomial given by the determinant of  $(A - \lambda I)$ .

1. **Basic eigenvalues and eigenvectors.** *Following the MATLAB cheatsheet, find the matrix  $V$  of eigenvectors and the diagonal matrix  $D$  of eigenvalues of one of the matrices from BTM problem (9.5.3), which is part of PS#18 (write the results next to the matrix):*

$$A = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix}$$

*You can verify that you have found eigenvectors by isolating the three eigenvectors  $V_1$ ,  $V_2$ , and  $V_3$  (see the cheatsheet!) and then calculating in turn  $AV_1$ ,  $AV_2$ , and  $AV_3$ . What do you expect to get? Carry out one or two of them in MATLAB to check.*

2. **Eigenvalues from the characteristic polynomial.** *Follow the cheatsheet to derive the characteristic polynomial for  $A$  using  $\det(A - \lambda I)$  and, if possible, compare to your calculation by hand of the determinant of  $(A - \lambda I)$ . Find the roots of the polynomial with MATLAB (using `roots`) and compare to the eigenvalues from the last part.*
3. **“Diagonalizing” a matrix.** *Using the matrix  $A$  once again and the matrix  $V$  of eigenvectors, calculate the matrix product  $V^{-1}AV$  (recall that the MATLAB command `inv(V)` is used to take an inverse). What do you get? Repeat the process for a random complex  $4 \times 4$  matrix  $A$  to see if it is a general result. What is the trace of the resulting matrix in terms of the eigenvalues of the original matrix? Now what is  $\text{Tr}(V^{-1}AV)$  based on our trace theorems? So conclude how  $\text{Tr}A$  is related to the eigenvalues. Repeat for the determinant.*

4. **BONUS: Eigenvalues of  $A^k$ .** Suppose we find the eigenvalues of a matrix  $A$ . Are the eigenvalues of  $A^2$ ,  $A^3$ , and so on related in an easy way? Let's do the numerical experiment. Use the matrix  $A$  from the first part. *Calculate the eigenvalues of  $A^k$  for  $k = 2$  and  $k = 3$  and then fill in the blank for the general rule: If  $\lambda$  is an eigenvalue of  $A$ , then \_\_\_\_\_ is an eigenvalue of  $A^k$ . Test your conjecture for  $k = 4$ .*
5. **BONUS: A neat identity.** One of my favorite matrix identities (and one I use frequently) states that for a matrix  $A$ ,

$$\det A = e^{\text{Tr} \ln A} .$$

Using the MATLAB functions (including `logm` for taking the logarithm of a matrix), try this out with a random *complex* matrix  $A$  to see if there are any obvious restrictions on the matrix. *Generate a  $4 \times 4$  matrix  $A$ , calculate  $\det A$  and then calculate  $e^{\text{Tr} \ln A}$ , and compare. Repeat a couple of times. Conclusion?* [Note: You will get a warning message for some matrices.]

### III. PS #18 Problem Using the arXiv E-Print Server (10 minutes)

[Note: This overlaps a lot with the bonus task from the last session.]

You can find the latest papers in physics (and also mathematics, non-linear science, computer science, and quantitative biology) by accessing the arXiv e-print server run by Cornell University Library. These papers are freely accessible (i.e., they don't cost anything and don't require access through a library) from any computer and in several different formats (including PDF).

1. Open a web browser and go to <http://arxiv.org/>, which is the main page of the arXiv. You'll see under the heading "Physics" many links to subfields such as "Astrophysics", "Condensed Matter" and so on.
2. Let's suppose we want to check out the brand-new papers in condensed matter. Find the "Condensed Matter" bullet and click on the "new" link. This will take you a page with abstracts (i.e., short descriptions) of all the papers uploaded the previous day. Each paper (which is generally called a "preprint" when it has not yet been published in a scientific journal) is identified by the archive name (for condensed matter it is **cond-mat**) and a unique number (which is built from the year, then the month, then a number that counts up from 1).

3. *Select a paper whose abstract looks interesting.* After each preprint number there are four links to choose from [abs, ps, pdf, other]. *Get a copy of the paper by clicking on pdf.* Once the paper has appeared in your PDF viewer you can read in online or print a copy (but not now!).
4. Now let's try a search. Return to the main page and find the bullet for "High Energy Physics – Lattice" (which is the `hep-lat` archive). Follow the "find" link. You will reach a web form page that can be used to fine tune a search. Let's do an easy one. *In the blank next to "Author(s)", type "kilcup" (without the quotes) and then click on the "Do Search" function.* You should get a list of papers by your 261–262 instructor, going back about 10 years (when the arXiv started). If you return to the Search form, you'll see that you can specify the year, the author, words in the title, and other things.
5. *Go ahead and do problem 5 of PS#18, which says: Find in the high-energy physics phenomenology archive the paper by Prof. Eric Braaten published in 2002 that has "condensate" in the title. For your answer to this problem, give the complete title of the paper, the full list of authors, and where the paper is published (i.e., the journal citation).*