Vectors

October 1, 2008

1 Scalars and Vectors

Scalars: A scalar quantity can be specified by just giving a *number*. Examples:

(a) The number n of balls in a box (e.g. $n = 5$)

(b) The mass m of a particle ($m = 3.1\ Kg$)

(c) The temperature T of a body ($T = 300\ ^oK$)

Vectors: A vector can be described by an arrow; it has a *magnitude* (the length of the arrow) and a *direction* (the direction of the arrow). Examples:

(a) The velocity $\vec{v}$ of a particle (e.g. $5\ m/s$ vertically upwards)

(b) The Force $\vec{F}$ on a body ($\vec{F} = 4\ N$ in the positive x direction)

(c) The electric field $\vec{E}$ at a point ($\vec{E} = 7\ Volts/m$ in the negative z direction)

We can describe a vector by choosing an orthonormal basis $\hat{x}, \hat{y}, \hat{z}$ of unit vectors. Then we can write

$$\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z} \tag{1}$$

Thus once we are given the orthonormal basis we can specify the vector by just giving three numbers $\{V_x, V_y, V_z\}$.

Changing the orthonormal basis:

The choice of orthonormal basis is not unique – we could choose a different set of orthogonal unit vectors $\hat{x}', \hat{y}', \hat{z}'$. Then we would write

$$\vec{V} = V'_x\hat{x}' + V'_y\hat{y}' + V'_z\hat{z}' \tag{2}$$

and the three numbers $\{V'^x, V'^y, V'^z\}$ would be different from the three numbers $\{V^x, V^y, V^z\}$.

Since we are talking about the same vector (i.e. the same 'arrow') in each case, we will call the vector $\vec{V}$ in each case. But since we used different axes in the two cases, the components were unprimed in the first case and denoted with primes in the second case. Clearly, we need to know a way to find the new components of $\vec{V}$ if we were given the old components of $\vec{V}$.

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Problem 1: Consider 2 dimensional space where a vector can be written as

$$\vec{V} = V_x\hat{x} + V_y\hat{y} \tag{3}$$

Now let the new axes $\hat{x}'\hat{y}'$ be obtained by rotating the old axes counterclockwise by an angle θ. Show that the components of $\vec{V}$ in the new frame are

$$\begin{aligned} V'_x &= \cos\theta V_x - \sin\theta V_y \\ V'_y &= \sin\theta V_x + \cos\theta V_y \end{aligned} \tag{4}$$

In other words, the new components can be given by the matrix rule

$$\begin{pmatrix} V'_x \\ V'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} \tag{5}$$

or in short

$$V' = RV \tag{6}$$

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The fact that the vector may have different components when seen in differnt frames may look like a tiresome irritant, but it is actually a simple example of a general idea (gauge invariance) that will underlie most of modern physics as we know it.

1.1 Operations with scalars and vectors

Vector → Scalar: Given a vector $\vec{V}$, can we make a scalar?

An incorrect choice: The vector was specified by three numbers $\{V_x, V_y, V_z\}$. Suppose we write

$$S = V_x \tag{7}$$

This looks likes a scalar to the extent that it is just one number, but it is not a 'good' or 'useful' way of getting a scalar from a vector. The reason is that if we change our orthonormal basis then the first component of the vector would change to V'_x. But a scalar was supposed to be free of any dependence on coordinate frames.

A correct choice: Write

$$S = (V_x)^2 + (V_y)^2 + (V_z)^2 \equiv \vec{V} \cdot \vec{V} \tag{8}$$

Even though the components of $\vec{V}$ change when we change the orthonormal frame, the value of S does *not* change. Thus this is indeed a good way of getting a scalar from a vector.

If we were given two vectors $\vec{V}, \vec{W}$ then we can make the scalars

$$\vec{V} \cdot \vec{V}, \quad \vec{W} \cdot \vec{W}, \quad \vec{V} \cdot \vec{W} \tag{9}$$

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Problem 2: Show that the dot product (8) is unchanged when we change the orthonormal basis.

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Vector → Vector:

Given a vector $\vec{V}$, can we make another vector from this in a 'good' way? 'Good' means that the result should not depend on any particular choice of orthonormal basis. Thus 'good' constructions are those which can be described in geometric terms, without specific reference to component values.

An incorrect example: Given a vector $\{V_x, V_y, V_z\}$, suppose we make $\vec{W} = \{2V_x, V_y, V_z\}$. This is not a geometric way of making a new vector; for example, if in some orthonormal basis we had $V_x = 0$, the $\vec{W} = \vec{V}$, while in other bases $\vec{W} \neq \vec{V}$.

Correct example I: Given $\vec{V}$, make $\vec{W} = 2\vec{V}$. Thus $\vec{W}$ has the same direction as $\vec{V}$ and twice the magnitude.

Correct example: II Suppose we are in 2-dimensions. Given a vector $\vec{V}$, we can make $\vec{W}$ to have the same length as $\vec{V}$ and be perpendicular to $\vec{V}$. (This still leaves two choices, and we can pick one by saying that $\vec{W}$ should be obtained by a $\frac{\pi}{2}$ anticlockwise rotation of $\vec{V}$.) In components

$$\vec{V} = \{V_x, V_y\} \ \rightarrow \ \vec{W} = \{-V_y, V_x\} \tag{10}$$

Correct example III: Now suppose we are in 3 dimensions. If we are given *two* vectors $\vec{V}, \vec{U}$ then we can make a new vector from their *cross product*:

$$\vec{W} = \vec{V} \times \vec{U} \tag{11}$$

Thus $\vec{W}$ is chosen to be perpendicular to both $\vec{V}, \vec{U}$, and its length is the area of the parallelogram formed by $\vec{V}, \vec{U}$:

$$|\vec{W}| = |\vec{V}||\vec{U}| \sin\theta \tag{12}$$

(θ is the angle between $\vec{V}, \vec{U}$.) This still leaves two choices for $\vec{W}$, and we choose one by adopting the right hand rule to fix the direction of $\vec{W}$.

In coordinates, we get

$$W_x = V_y U_z = V_z U_y, \quad W_y = V_z U_x - V_x U_z, \quad W_z = V_x U_y - U_y U_x \tag{13}$$

Note that Example III is similar to Example II in that $\vec{W}$ is orthogonal to the given vectors ($\vec{V}$ in the first case, $\vec{V}, \vec{U}$ in the second). If we were in 4 dimensions, we could make a vector $\vec{W}$ that was orthogonal to *three* given vectors $\vec{V}, \vec{U}, \vec{X}$. We will see this construction later; it is a generalization if the cross product.

2 Scalar fields

Scalar field: We get a *scalar field* if we attach a value for the scalar to each point in space. Examples:

(a) The temperature at each point of space is given by a number $T(x, y, z)$.

(b) The air in the room is described by the *density* of molecules at each point: $\rho(x, y, z)$ molecules per cubic cm at the location (x, y, z).

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Problem: Write a scalar field $h(x, y)$ that describes the height h of a hill at the point x, y on the ground.

Problem: Write a height function $h(x, y)$ that describes an infinitely long straight valley, lying between two hills.

Problem: Write a scalar field that describes the temperature of the earth. The temperature is highest at the center of the earth, and drops off to absolute zero as we go further and further out into the atmosphere. Also, It is colder at the poles than at the equator.

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Example: Let the space be the surface of a sphere, spanned by coordinates θ, ϕ. An example of a scalar field is

$$T(\theta, \phi) = \sin^2\theta \cos\phi \tag{14}$$

We would *not* get a scalar field by writing

$$T(\theta, \phi) = \sin^2\theta \cos(\frac{5\phi}{4}) \tag{15}$$

This is because on the sphere the point (θ, ϕ) is the same as the point $(\theta, \phi + 2\pi)$. But for (15) we get

$$T(\theta, \phi) \neq T(\theta, \phi + 2\pi) \tag{16}$$

So we have not assigned a unique value for the scalar T for each point, and thus have not given a proper scalar field.

3 Vector fields

We get a *vector field* if we attach a vector to each point of our space.

Example: On the space (x, y, z) we can write a vector field as

$$\vec{V} = V_x(x, y, z)\hat{x} + V_y(x, y, z)\hat{y} + V_z(x, y, z)\hat{z} \tag{17}$$

We can choose any functions for V_x, V_y, V_z, for example

$$V_x = x^2 y, \quad V_y = \tanh(x + z), \quad V_z = \sin y \tag{18}$$

3.1 Vector field on a surface

Our space can be a 2-dimensional surface $\mathcal{S}$, rather than the entire 3-dimensional space x, y, z. If we think in terms of this surface being our entire world, then any vector on this surface must point along the surface; i.e., it must be tangent to the surface. For example, if the vector described the velocity of a particle living on the surface, then there is no way this velocity can point 'out' of the surface.

To write a vector field on $\mathcal{S}$ we can to choose a set of orthogonal unit vectors at each point of the surface $\mathcal{S}$. For a 2-dimensional surface, there will be two such unit vectors.

Example: Let us write a vector field that can describe wind direction on the surface of the earth.

Choose unit vectors along the θ and ϕ directions, and write

$$\vec{V} = V_\theta(\theta, \phi)\hat{\theta} + V_\phi(\theta, \phi)\hat{\phi} \tag{19}$$

Take for example

$$V_\theta(\theta, \phi) = \sin\theta, \quad V_\phi(\theta, \phi) = \cos\theta\cos\phi \tag{20}$$

Note that the choice of basis vectors is not good at two points of the sphere: the North pole and the South pole. At the North pole we have $\theta = 0$, but all values of ϕ give us the *same point*. So we cannot draw a unit vector that points along the $\hat{\phi}$ direction. The same happens at the South pole. Nevertheless the basis vectors $\hat{\theta}, \hat{\phi}$ are commonly used. Since the vector field varies continuously from point to point, we can easily understand its magnitude and direction at the poles by looking at the vectors attached to points *near* these poles.

4 Making new fields from old

Given a scalar, there is no geometric way to get a vector from it. But we will see that we can take a scalar *field* and make a vector *field* from it in a geometric way. As before, the word 'geometric'; means that the answer does not depend on any choice of basis vectors that we might have used; two people using two different sets of basis vectors will still arrive at the same final vector field. In this section we study this operation of making vector fields from scalar fields, and also operations that take us from vector fields to scalar fields, and vector fields to vector fields.

4.1 Making vector fields from scalar fields

Given a scalar field $S(x, y, z)$ we can write a vector field

$$\vec{V} = \frac{\partial S}{\partial x}(x, y, z)\hat{x} + \frac{\partial S}{\partial y}(x, y, z)\hat{y} + \frac{\partial S}{\partial z}(x, y, z)\hat{z} \tag{21}$$

This vector field is called the *gradient* of the scalar field S, and is written as $\vec{\nabla} S$.

While $\vec{\nabla} S$ *looks* like a vector field because it has the required three components, we need to *show* that these three components indeed behave like the three components of a vector.

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Problem: Prove that the components of $\vec{\nabla} S$ transform like a vector.

Solution:

(i) First let us work at the origin $(0, 0, 0)$; we can later shift our origin to any other point, and so will understand the behavior at all other points as well.

(ii) The components of $\vec{\nabla} S$ at the origin are

$$\{\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}\} \tag{22}$$

where all derivatives are evaluated at $(0,0,0)$.

(iii) Now let us study the rotation of coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{23}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R^{-1} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \tag{24}$$

Explicitly, we have

$$x = (R^{-1})_{11}x' + (R^{-1})_{12}y' + (R^{-1})_{13}z' \tag{25}$$

and similarly for y, z. Recall that $R^{-1} = R^T$. Thus

$$x = (R^T)_{11}x' + (R^T)_{12}y' + (R^T)_{13}z' \tag{26}$$

or using the definition of transpose

$$x = (R)_{11}x' + (R)_{21}y' + (R)_{31}z' \tag{27}$$

Similarly,

$$y = (R)_{12}x' + (R)_{22}y' + (R)_{32}z' \tag{28}$$

$$z = (R)_{13}x' + (R)_{23}y' + (R)_{33}z' \tag{29}$$

(iv) In the new coordinates x', y', z' obtained after rotation, the components of $\vec{\nabla}S$ are

$$\{\frac{\partial S}{\partial x'}, \frac{\partial S}{\partial y'}, \frac{\partial S}{\partial z'}\} \tag{30}$$

(v) By the chain rule for derivatives we find

$$\frac{\partial S}{\partial x'} = \frac{\partial S}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial S}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial S}{\partial z}\frac{\partial z}{\partial x'} \tag{31}$$

From (27)-(29) we get

$$\frac{\partial x}{\partial x'} = (R)_{11}, \quad \frac{\partial y}{\partial x'} = (R)_{12}, \quad \frac{\partial z}{\partial x'} = (R)_{13} \tag{32}$$

Thus

$$\frac{\partial S}{\partial x'} = (R)_{11}\frac{\partial S}{\partial x} + (R)_{12}\frac{\partial S}{\partial y} + (R)_{13}\frac{\partial S}{\partial z} \tag{33}$$

(vi) Doing similar computations for $\frac{\partial S}{\partial y'}, \frac{\partial S}{\partial z'}$, we find

$$\begin{pmatrix} \frac{\partial S}{\partial x'} \\ \frac{\partial S}{\partial y'} \\ \frac{\partial S}{\partial z'} \end{pmatrix} = R \begin{pmatrix} \frac{\partial S}{\partial x} \\ \frac{\partial S}{\partial y} \\ \frac{\partial S}{\partial z} \end{pmatrix} \tag{34}$$

This shows that the components of $\vec{\nabla}S$ change exactly as required for the components of a vector.

Problem: Show that the three functions

$$\frac{\partial^2 S}{\partial x^2}(x,y,z), \quad \frac{\partial^2 S}{\partial y^2}(x,y,z), \quad \frac{\partial^2 S}{\partial z^2}(x,y,z) \tag{35}$$

do *not* transform like the three components of a vector.

Problem: Find the gradient of the following scalar fields:

(a) $S(x,y,z) = x^2 + y^2 + z^2$

(b) $T(x,y,z) = x \sin y + y \cos x$

Problem: The height of a hill is given by

$$h(x,y) = 1 - \tanh(x^2 + y^2 + 2x + y + 3) \tag{36}$$

Find the point (x,y) where the hill is highest.

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4.2 Making scalar fields from vector fields

From a vector field V we can make a scalar field

$$\vec{\nabla} \cdot \vec{V}(x,y,z) = \frac{\partial V_x(x,y,z)}{\partial x} + \frac{\partial V_y(x,y,z)}{\partial y} + \frac{\partial V_z(x,y,z)}{\partial z} \tag{37}$$

Problem:

(a) Prove that $\vec{\nabla} \cdot \vec{V}(x,y,z)$ is a scalar.

(b) Prove that

$$\frac{\partial^2 V_x(x,y,z)}{\partial x^2} + \frac{\partial^2 V_y(x,y,z)}{\partial y^2} + \frac{\partial^2 V_z(x,y,z)}{\partial z^2} \tag{38}$$

is *not* a scalar.

4.3 Making a vector field from a vector field

Given a vector field $\vec{V}$ we can make another vector field by taking the curl:

$$\vec{W} = \vec{\nabla} \times \vec{V} \tag{39}$$

In components, we have

$$\begin{aligned} W_x &= \frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x} \\ W_y &= \frac{\partial V_y}{\partial z} - \frac{\partial V_z}{\partial y} \\ W_z &= \frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} \end{aligned} \tag{40}$$

4.3.1 Understanding the curl

To understand the curl operation better, let us name the above components slightly differently:

(a) The $x - y$ component: $\frac{\partial V_x}{\partial V_y} - \frac{\partial V_y}{\partial V_x}$

(b) The $y - z$ component: $\frac{\partial V_y}{\partial V_z} - \frac{\partial V_z}{\partial V_y}$

(c) The $z - x$ component: $\frac{\partial V_z}{\partial V_x} - \frac{\partial V_x}{\partial V_z}$

We also have

$$y - x \text{ component} : \frac{\partial V_y}{\partial V_x} - \frac{\partial V_x}{\partial V_y} = - [x - y \text{ component}] \tag{41}$$

and so on. We can now see that the standard notation (40) names these components in the following way. If we have the $x - y$ component of curl, we just label it by the index which is *not* present in the $x - y$ label; namely the index z. Thus instead of having to use *two* indices $x - y$ to label the component, we just have to write *one* index z. Of course we have to decide if we will let this z component correspond to the $x - y$ component or the $y - x$ component; as we saw in (41) these two differ by a sign. The choice is settled by taking once for all a cyclic order for the indices $x - y - z$. Then the $x - y$ component will be called z, the $y - z$ component will be called x and the $z - x$ component will be called y. This cyclic choice $x - y - z$ corresponds to the 'right hand rule' in computing the cross product in (39). We could have taken the other cyclic order $x - z - y$, and then we would have a 'left hand rule'. Since these are just naming conventions, we could take either rule as long as we follow it consistently each time we have to write a cross product.

5 Integrating fields

We have seen operations that differentiate scalar or vector fields, namely $\vec{\nabla}S, \vec{\nabla}\cdot\vec{V}$. Now we will look at operations where we *integrate* scalar and vector fields.

5.1 Integrating a scalar field

Consider a scalar field $S(x, y, z)$. Consider an integral

$$I = \int_D dxdydz\ S(x, y, z) \tag{42}$$

where D is some region of space over which we want to compute the integral. In pictures, the integration corresponds to doing the following steps:

(a) Cut up the region D into little cubes.

(b) Consider one of these cubes. Suppose it is centered at the point (x', y', z'). Find the volume δV of the cube.

(c) Find the value of S in the cube. Since the cube is small, it does not matter exactly where in the cube we measure S; for example we can take $S(x', y', z')$ at one corner of the cube or at the midpoint of the cube.

(d) The contribution of this cube to the integral is

$$\delta I = S(x', y', z')\delta V \tag{43}$$

(e) Add over all cubes in the region D to get the integral I.

Example: Integrate $S = x^2y + 2z$ over the region

$$-1 \le x \le 1, \quad 0 \le y \le 4, \quad -2 \le z \le 0 \tag{44}$$

Solution: Consider a point (x', y', z') in the region of integration. Around this point let a small cube be made as

$$x' \le x \le x' + dx', \quad y' \le y \le y' + dy', \quad z' \le z \le z' + dz' \tag{45}$$

The volume of this cube is $\delta V = dx'dy'dz'$. We compute S for this cube at the corner (x', y', z'), getting

$$S(x', y', z') = x'^2y' + 2z' \tag{46}$$

Thus the contribution of this cube to the integral is

$$\delta I = S(x', y', z')\delta V = (x'^2y' + 2z')dx'dy'dz' \tag{47}$$

The full integral is then obtained by summing over the cubes.

$$I = \int_{x'=-1}^{1} \int_{y'=0}^{4} \int_{z'=-2}^{0} (x'^2 y' + 2z') dx' dy' dz' \tag{48}$$

Example: Suppose the population density on the earth is

$$\sigma(\theta, \phi) = a \cos\theta \sin\phi \tag{49}$$

Find the number of people in the region $0 \le \theta \le \frac{\pi}{2}$, $\frac{\pi}{2} \le \phi \le \pi$. (Assume that the radius of the earth is R.)

Solution: Consider a point θ', ϕ' on the surface of the earth. Take a little plaquette around this point given by

$$\theta' \le \theta \le \theta' + d\theta, \quad \phi' \le \phi \le \phi' + d\phi' \tag{50}$$

The area of this plaquette is $dA = (R d\theta')(R \sin\theta' d\phi')$. The number of people in this plaquette is

$$\sigma dA = aR^2 \cos\theta' \sin\theta' \sin\phi' d\theta' d\phi' \tag{51}$$

The number of people in the given region is then

$$N = \int_{\theta'=0}^{\frac{\pi}{2}} \int_{\phi'=\frac{\pi}{2}}^{\pi} aR^2 \cos\theta' \sin\theta' \sin\phi' d\theta' d\phi' \tag{52}$$

5.2 Integrating vector fields

Recall that integration is essentially a process of *addition*. A principal point to note is: *we do not add vectors at different points.* We must first make a scalar out of the vector at each point, and then add these scalars.

There can be three kinds of integrals:

(a) **Integral over a 1-dimensional** *curve C*: We take the dot product of $\vec{V}$ with the *tangent* to the curve; this gives a scalar. We then add these scalar contributions from different points of the curve.

(b) **Integral over a 2-dimensional surface:** We take the dot product of $\vec{V}$ with the *normal* to the surface, getting a scalar. We then add these scalar contributions from different points of the surface.

(c) **Integral over a 3-dimensional** *volume.* This time we have no geometric quantity to dot $\vec{V}$ with. We can however dot $\vec{V}$ with *itself*, getting a scalar

$$S(x, y, z) = (\vec{V} \cdot \vec{V})(x, y, z) \tag{53}$$

We can then integrate this scalar field

$$I = \int_V dxdydz\ S(x, y, z) \tag{54}$$

But unlike (a), (b) above, this quantity is not linear in $\vec{V}$; instead it is *quadratic* since we have used $V \cdot V$ and not $\vec{V}$ itself.

5.3 Integrating a vector field along a curve

Suppose we are given a curve C and a vector field $\vec{V}(x, y, z)$. Speaking geometrically, we integrate $\vec{V}$ along C by performing the following steps:

(a) Cut up the curve into small segments. Take one such segment. Draw an arrow from the starting point of the segment to the ending point of the segment. This arrow is a small vector, which we call $\vec{dl}$. Since the segment is small, $\vec{dl}$ lies almost along the segment.

(b) The vector field gives a vector $\vec{V}$ for each point in our space. Take the vector at some point of the chosen segment. Since the segment is small, it will not matter which point of the segment we choose; for example we can take the starting point of the segment, or the ending point, or the midpoint.

(c) Take the dot product of this vector with the vector $\vec{dl}$. This gives a scalar

$$\delta I = \vec{V} \cdot \vec{dl} \tag{55}$$

This can also be written as

$$\delta I = \vec{v} \cdot \hat{t} dl \tag{56}$$

where $\hat{t}$ is the tangent to the curve C at the chosen point on C, and dl is the length of the segment.

(d) Add these scalars over all the different segments of the curve C. The result is

$$I = \int_C \vec{V} \cdot dl \tag{57}$$

Application: If we have a force $\vec{F}$ and its point of application moves by an amount $\vec{dl}$ then the work done by the force is given by

$$dW = \vec{F} \cdot \vec{dl} \tag{58}$$

Overall, if the force pushes a body along a curve C then the work done is

$$W = \int_C \vec{F} \cdot \vec{dl} \tag{59}$$

Example: Integrate the vector field

$$\vec{V} = x^2\hat{x} + y^2\hat{y} + z^2\hat{z} \tag{60}$$

over the straight line from $(0,0,0)$ to $(3,4,5)$.

Solution: Points on this curve are given by $(3\alpha, 4\alpha, 5\alpha)$ with $0 \le \alpha \le 1$. Take the segment that stretches between the points given by

$$(3\alpha', 4\alpha', 5\alpha') \quad and \quad (3(\alpha' + d\alpha'), 4(\alpha' + d\alpha'), 5(\alpha' + d\alpha')) \tag{61}$$

The vector between the endpoints of this segment is

$$\vec{dl} = (3d\alpha', 4d\alpha', 5d\alpha') = d\alpha'(3,4,5) \tag{62}$$

The vector field at the point $(3\alpha', 4\alpha', 5\alpha')$ is

$$\vec{V} = (9\alpha'^2, 16\alpha'^2, 15\alpha'^2) \tag{63}$$

Thus the contribution to the line integral from this segment is

$$dI = \vec{V} \cdot \vec{dl} = d\alpha' \alpha'^2 (26 + 64 + 125) = 215\alpha'^2 d\alpha' \tag{64}$$

The full integral is then

$$I = \int_{\alpha'=0}^{1} 215\alpha'^2 d\alpha' = \frac{215}{3} \tag{65}$$

5.3.1 Parametrizing curves

How do we describe a curve? First, we choose a parameter s to label different points on the curve. This parameter will increase from one end of the curve to the other, and we can specify any point on the curve by giving the value of s for that point. Next, for the point on the curve given by a value s, we should know where in space this point sits; i.e. we give its location $(x(s), y(s), z(s))$ We should also specify the range of the parameter s so that we know where the curve begins and where it ends. Thus a curve is specified by giving

$$(x(s), y(s), z(s)), \qquad s_i \le s \le s_f \tag{66}$$

Example: Integrate the vector field

$$\vec{V} = y\hat{x} - x\hat{y} + 2\hat{z} \tag{67}$$

along a circle of radius R around the origin circle of the $x - y$ plane.

Solution: The circle is given by

$$x^2 + y^2 = R^2, \quad z = 0 \tag{68}$$

Let the parameter along the circle be the angular coordinate θ. Then

$$x(\theta) = R\cos\theta, \quad y(\theta) = R\sin\theta, \quad z(\theta) = 0, \quad 0 \le \theta < 2\pi \tag{69}$$

Consider a small segment of the curve lying between θ and $\theta + d\theta$. The arrow between these endpoints is a vector $\vec{dl}$. The x component of $\vec{dl}$ is the change in x between the endpoints, and so on. Thus $\vec{dl}$ has components

$$\begin{aligned} dl_x &= \delta x = \frac{dx(\theta)}{d\theta} d\theta = -R \sin\theta d\theta \\ dl_y &= \delta y = \frac{dy(\theta)}{d\theta} d\theta = R \cos\theta d\theta \\ dl_z &= \delta z = \frac{dz(\theta)}{d\theta} d\theta = 0 \end{aligned} \tag{70}$$

Thus

$$dI = \vec{V} \cdot \vec{dl} = [-R \sin\theta y - R \cos\theta x] d\theta = -R^2 (\sin^2\theta + \cos^2\theta) d\theta = -R^2 d\theta \tag{71}$$

and

$$I = \int_{\theta=0}^{2\pi} (-R^2 d\theta) = -2\pi R^2 \tag{72}$$

5.4 Integrating a vector field over a surface

Suppose we are given a surface $\mathcal{S}$, and a vector field $\vec{V}(x, y, z)$. We can integrate $\vec{V}$ over $\mathcal{S}$ by the following steps:

(i) Cut up the surface $\mathcal{S}$ into little plaquettes. Take one such plaquette. Let its area be dA.

(ii) Take the vector field at any point on this little plaquette. Since the plaquette is small, it will not matter which point we take; for example we can take one corner of the plaquette or the midpoint of the plaquette. Let the chosen point be (x', y', z'). Let $\vec{V}(x', y', z')$ be the value of the vector field at this point.

(iii) We cannot add vectors at different points, so we have to make a scalar out of $\vec{V}(x', y', z')$. In the case of the line integral, we could do a dot product with the vector along the curve. The surface, however, has an infinite number of tangential directions, and there is no way to choose a particular one. But the plaquette has a well defined direction *normal* to it. We will take the unit vector $\hat{n}$ along this normal direction. Actually there are two directions for this unit normal, and we will have to be told which one to take. Once we choose a given one for one plaquette of $\mathcal{S}$ we will be able to say which one we need for all other plaquettes by continuity.

(iv) Thus compute the quantity

$$dI = (\vec{V}(x', y', z') \cdot \hat{n}) dA \tag{73}$$

(v) Add such contributions from all plaquettes to get the overall integral

$$I = \int_{\mathcal{S}} (\vec{V} \cdot \hat{n}) \; dA \tag{74}$$

6 'Cancelling' derivatives and integrals

In a rough sense derivatives and integral are 'opposites' of each other, and so should 'cancel'. In ordinary algebra, we have the relation

$$\int_{x=a}^{x=b} dx\, \frac{df(x)}{dx} = f(b) - f(a) \tag{75}$$

Thus the derivative on f has 'cancelled' the integral operation, and we are back to just f, with no derivatives or integrals. But where do we compute this function f? At the two *endpoints* of the integration region, which can be considered the *boundary* of the integration region.

We will now see how this phenomenon generalizes to scalar and vector fields. We can have three kinds of integrals:

(a) An integral $\int_C$ over a curve. This is just a 'single' integral like the integral $\int dx$ in (75).

(b) An integral $\int\int_{\mathcal{S}}$ over a surface. This is a *double* integral like $\int\int dxdy$, since there are two directions in the surface.

(c) An integral $\int\int\int_V$ over a volume. This is a triple integral, like $\int\int\int dxdydz$.

On the other hand the derivatives involved in

$$\vec{\nabla} S, \quad \vec{\nabla}\cdot\vec{V}, \quad \vec{\nabla}\times\vec{V} \tag{76}$$

are all 'single' derivatives. For example, even if we add $\frac{\partial V_x}{\partial x}$ to $\frac{\partial V_y}{\partial y}$, we still have in each term just one operation of differentiation. (A double derivative would have terms like $\frac{\partial^2 V_x}{\partial x \partial y}$.)

This counting is important because 'one derivative is cancelled by one integral'. Suppose we integrate $\vec{\nabla}\times\vec{V}$ over a surface. Then the derivative on $\vec{V}$ will cancel one of the two integrals over the surface, and we will be left with one integral over $\vec{V}$. From what we saw in (76), this integral should be over a 1-dimensional *boundary* of the surface $\mathcal{S}$. In this way it is very easy to remember the integration theorems for scalar and vector fields.

6.1 Starting with a scalar field

(i) Start with a scalar field $S(x, y, z)$.

(ii) Apply a derivative; this will be a gradient operation giving a vector field

$$\vec{V}(x, y, z) = \vec{\nabla} S(x, y, z) \tag{77}$$

(iii) Integrate this vector field along a *curve* C. Let the starting point of the curve be (x_i, y_i, z_i), and the ending point be (x_f, y_f, z_f). We get a *number*

$$I = \int_C \vec{V}\cdot\vec{dl} \tag{78}$$

(iv) The theorem says that the derivative and integral will cancel, giving

$$I = \int_C \vec{V} \cdot \vec{dl} = S(x_f, y_f, z_f) - S(x_i, y_i, z_i) \tag{79}$$

Note that we are left with the scalar field values at the endpoints of the curve. which are the *boundary* of the curve.

Proof of (79):

(a) The vector $\vec{V}(x', y', z')$ is

$$\vec{V} = \frac{\partial S}{\partial x'}(x', y', z') + \frac{\partial S}{\partial y'}(x', y', z') + \frac{\partial S}{\partial z'}(x', y', z') \tag{80}$$

(b) Cut up the curve C into little segments. Let the segment under consideration go from (x', y', z') to $(x' + dx', y' + dy', z' + dz')$. The vector $\vec{dl}$ is

$$(dx', dy', dz') \tag{81}$$

(c) Thus

$$\frac{\partial S}{\partial x'}dx' + \frac{\partial S}{\partial y'}dy' + \frac{\partial S}{\partial z'}dz' = S(x' + dx', y' + dy', z' + dz') - S(x', y', z') \tag{82}$$

(d) Let us add over all little segments. Suppose the first segment goes from $\vec{r}_i$ to $\vec{r}_1$, the next from $\vec{r}_1$ to $\vec{r}_2$, and so on till the last one from $\vec{r}_n$ to $\vec{r}_f$. Then the sum of the contributions (82) gives

$$(S_{\vec{r}_1} - S_{\vec{r}_i}) + (S_{\vec{r}_2} - S_{\vec{r}_1}) + (S_{\vec{r}_3} - S_{\vec{r}_2}) + \ldots + (S_{\vec{r}_f} - S_{\vec{r}_n}) = S_{\vec{r}_f} - S_{\vec{r}_i} \tag{83}$$

This proves the theorem.

6.1.1 Checking other possibilities for starting with a scalar

Starting with the scalar, the derivative can only give $\vec{\nabla} S$. Above we had integrated $\vec{\nabla} S$ along a curve. What happens if we integrate it over a surface

$$\int\int_{\mathcal{S}} dA \ (\vec{\nabla} S \cdot \hat{n}) \tag{84}$$

There is no nice cancellation of the derivative against an integral this time. To see this, note that if we did have such a cancellation, then we would be left with one integral of S over the boundary of $\mathcal{S}$, which is a curve C. But there is no geometric way to integrate a scalar over a curve.

If we wish to integrate over a volume R, then we have to make a scalar out of $\vec{\nabla} S$ by dotting it with itself

$$I = \int\int\int_R dxdydz \ = (\vec{\nabla} S \cdot \vec{\nabla} S) \tag{85}$$

But this does not have the form of an integral followed by a derivative, and there will again be no ‘cancellation’ theorem in this case. An expression like (85) does arise in physic as the *Lagrangian density* for scalar fields later.

6.2 Starting with a vector field

Now let us start with a vector field $\vec{V}(x, y, z)$. We wish to take a derivative of this vector field. We can do two things: we can compute the divergence $\vec{\nabla} \cdot \vec{V}$ or take the curl $\vec{\nabla} \times \vec{V}$.

6.2.1 Integrating a divergence

(i) Let us first consider taking $\vec{\nabla} \cdot \vec{V}$. This is a scalar field. We can integrate a scalar field over a region which is a *volume*

$$I = \int_R dxdydz \vec{\nabla} \cdot \vec{V}(x, y, z) \tag{86}$$

(ii) Here we have an integral and a derivative, so we expect that we can cancel these two operations and get some quantity involving just $\vec{V}$ on the boundary of R.

(iii) The boundary of the volume R will be a 2-dimensional *surface* $\mathcal{S}$. We can write the surface integral

$$\int_{\mathcal{S}} (\vec{V} \cdot \hat{n}) dA \tag{87}$$

We still have to choose one of the two allowed directions for the normal $\hat{n}$. We will need the normal which points *ourwards* from the region R.

(iv) With all this, we can write the theorem

$$\int_R dxdydz \vec{\nabla} \cdot \vec{V}(x, y, z) = \int_{\mathcal{S}} (\vec{V} \cdot \hat{n}) dA \tag{88}$$

6.2.2 Integrating a curl

Suppose we have a vector field $\vec{V}(x, y, z)$ and have taken its curl $\vec{\nabla} \times \vec{V}$. We can integrate this curl over a *2-dimensional surface* $\mathcal{S}$.

$$\int_{\mathcal{S}} (\vec{\nabla} \times \vec{V}) \cdot \hat{n} \tag{89}$$

The surface integral is a 'double intergral'. We can cancel one integral against the derivative, so we should be left with a single integral using $\vec{V}$. Is there a natural guess for the answer? We can take a line integral

$$\int_C \vec{V} \cdot \vec{dl} \tag{90}$$

We have the following theorem (Stokes theorem)

$$\int_{\mathcal{S}} (\vec{\nabla} \times \vec{V}) \cdot \hat{n} = \int_C \vec{V} \cdot \vec{dl} \tag{91}$$

where C is the *boundary* of the surface $\mathcal{S}$.

Problem: Prove (**??**).

7 Two derivatives

We have studied scalar fields S and vector fields $\vec{V}$, and then studied the action of derivative operators on these fields: $\vec{\nabla}S, \vec{\nabla}\cdot\vec{V}, \vec{\nabla}\times\vec{V}$. Now we look at operations where *two* derivatives are applied to the field.

We have already seen in many cases that applying two derivatives leads to a quantity that is not a scalar or a vector, and thus of no general interest to us. Let us now note the cases where we *do* get a geometric quantity by applying two derivatives.

7.1 Staring with a scalar

The first derivative of S must be a gradient

$$\vec{\nabla}S = \frac{\partial S}{\partial x}(x,y,z)\hat{x} + \frac{\partial S}{\partial y}(x,y,z)\hat{y} + \frac{\partial S}{\partial z}(x,y,z)\hat{z} \tag{92}$$

The second derivative can be a divergence or a curl:

(a)

$$\vec{\nabla}\cdot\vec{\nabla}S = \frac{\partial^2 S}{\partial x^2}(x,y,z) + \frac{\partial^2 S}{\partial y^2}(x,y,z) + \frac{\partial^2 S}{\partial z^2}(x,y,z) \equiv \triangle S \tag{93}$$

The scalar field $\triangle S$ thus obtained is called the Laplacian of S.

(b) $\vec{\nabla}\times\vec{\nabla}S$: For the x component of this vector we have

$$\frac{\partial}{\partial y}\frac{\partial S}{\partial z}(x,y,z) - \frac{\partial}{\partial z}\frac{\partial S}{\partial y}(x,y,z) = 0 \tag{94}$$

and similarly for the other components. Thus we find that

$$\vec{\nabla}\times\vec{\nabla}S = 0 \tag{95}$$

or in words 'curl of grad is zero'. One can remember this rule by noting that (95) has the cross product of the two $\vec{\nabla}$ operations. If these were two ordinary vectors, the cross product would vanish.

7.2 Starting with a vector

There are again two possibilities:

(a) $\vec{\nabla}(\vec{\nabla}\cdot\vec{V})$

(b) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V})$: This equals

$$\frac{\partial}{\partial x}(\frac{\partial V_y}{\partial z} - \frac{\partial V_z}{\partial y}) + \frac{\partial}{\partial y}(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}) + \frac{\partial}{\partial z}(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}) = 0 \tag{96}$$

Thus

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0 \tag{97}$$

or in words, 'divergence of curl is zero'. Again, we can remember this rule by noting that if the two $\vec{\nabla}$ operations were ordinary vectors, this expression would vanish.