Name:

656 Exercise 1 (based on Pollock-Stump 8.15) Wed Jan 11, 2012 (Not Graded)

Consider a circular loop of wire in the $x-y$ plane, with radius a, centered at the origin, carrying a current I.

(a) Find $\vec{B}$ at the center of the loop

(b) Find $\vec{B}(z)$ all along the z axis

Let the current in the loop run counterclockwise as seen from above. By the Biot-Savart law we have

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{dl}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\int \frac{\vec{dl}\times\vec{r}}{r^3} \tag{1}$$

We have

$$\vec{dl} = a d\phi\hat{\phi} \tag{2}$$

$$\vec{r} = z\hat{z} - a\hat{r}, \quad r = (z^2+a^2)^{\frac{1}{2}} \tag{3}$$

$$\vec{dl}\times\vec{r} = a d\phi(z\hat{r} + a\hat{z}) \tag{4}$$

By symmetry only the $\hat{z}$ part will remain. We will get

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} a d\phi \frac{a}{(z^2+a^2)^{\frac{3}{2}}}\ \hat{z} = \frac{\mu_0 I}{2}\frac{a^2}{(z^2+a^2)^{\frac{3}{2}}}\hat{z} \tag{5}$$

(c) Compute $\int_{-\infty}^{\infty} B_z(z)dz$ along the z axis. Could you have deduced the value of this integral from Ampere's law?

$$\int_{-\infty}^{\infty} B_z dz = \frac{\mu_0 I}{2}\int_{-\infty}^{\infty}\frac{a^2}{(z^2+a^2)^{\frac{3}{2}}}dz \tag{6}$$

Let $z = a\sinh\mu$, $dz = a\cosh\mu d\mu$, $z^2+a^2 = a^2\cosh^2\mu$,

$$\int_{-\infty}^{\infty} B_z dz = \frac{\mu_0 I}{2}\int_{\mu=-\infty}^{\infty}\mathrm{sech}^2\mu \mathrm{d}\mu = \frac{\mu_0 \mathrm{I}}{2}\tanh\mu|_{-\infty}^{\infty} = \mu_0\mathrm{I} \tag{7}$$