1 The action for particles in special relativity

In Newtonian mechanics we have seen that the Lagrangian is a function of of the generalized coordinates q_k, their time derivatives $\dot{q}_k$ and possibly the time t as well. Thus we write $L[q_k, \dot{q}_k, t]$, where $\dot{q}_k \equiv \frac{dq_k}{dt}$. For the simple example of a particle moving in a potential well, the Lagrangian is just

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x) \tag{1}$$

Now consider a particle of mass m, with no potential, but now including the effects of special relativity. The distance in usual space is defined by

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{2}$$

but when we include time as another coordinate, we define distance as

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{3}$$

Here and in all that follows we will choose units so that $c = 1$. The variational principle for a free particle in special relativity is very simple: we just extremize the path length given by the definition of distance (3). Thus

$$S = -m\int_i^f ds = -m\int_i^f \sqrt{dt^2 - dx^2 - dy^2 - dz^2} \tag{4}$$

Here the constant $-m$ will make no difference to the extremization problem, but it makes the low velocity limit of (5) agree with the Newtonian Lagrangian

$$\begin{aligned} S &= -m\int_i^f ds = -m\int_i^f \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = -m\int_i^f \sqrt{1 - (\frac{dx}{dt})^2 - (\frac{dy}{dt})^2 - (\frac{dz}{dt})^2} \ dt \\ &\approx -m\int_i^f [1 - \frac{1}{2}|\vec{v}|^2]dt = -m(t_f - t_i) + \int_i^f \frac{1}{2}m|\vec{v}|^2 dt \end{aligned} \tag{5}$$

where in the second line we have used the binomial expansion for low velocities (i.e. velocities much smaller than 1, which in our units is the speed of light). Now note that the part $-m(t_f - t_i)$ is the same for all paths in the variational principle when the initial and final endpoints are held fixed. So we may drop this term from the variational principle, and we just get the kinetic energy T, as expected for the Newtonian limit for a free particle.

In the first line of (5) we extracted dt from under the square root, and wrote our coordinates as $x(t), y(t), z(t)$. While this is a correct procedure, one may wish to keep complete symmetry between all four coordinates x, y, z, t, since this would be more in keeping with the spirit of relativity. What will then be the variable of integration in the action in place of t? We choose an arbitrary parameter τ to label the points on the worldline of the particle. Then the worldline is described by the variables

$$t(\tau), x(\tau), y(\tau), z(\tau) \tag{6}$$

The action can then be written as

$$S = -m \int_i^f \sqrt{(\frac{dt(\tau)}{d\tau})^2 - (\frac{dx(\tau)}{d\tau})^2 - (\frac{dy(\tau)}{d\tau})^2 - (\frac{dz(\tau)}{d\tau})^2} \; d\tau \tag{7}$$

This is just like a usual Lagrangian problem, except that now τ acts like our 'time', and there are *four* coordinates instead of three.

2 General coordinates

Suppose we use polar coordinates instead of Cartesian. Then the distance between two infinitesimally separated points is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{8}$$

instead of (2). If we include time, then we would get

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{9}$$

instead of (3).

Now suppose we have a gravitational field, created by some heavy mass like the sun. By Einstein's theory of general relativity, we can absorb the entire effect of gravity by just replacing 'flat spacetime' by 'curved spacetime'. For a spherical source like the sun, the spacetime outside the source is given by the following distance measure (called the Schwarzschild metric)

$$ds^2 = (1 - \frac{2GM}{r})dt^2 - \frac{1}{\frac{1-2GM}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{10}$$

There is a convenient shorthand for writing such distance relations. We write

$$ds^2 = \sum_{a=1}^{d} \sum_{b=1}^{d} g_{ab}(\xi^1, \dots \xi^d) d\xi^a d\xi^b \tag{11}$$

where $\xi^a, a = 1, 2, \dots d$ are the coordinates and $g_{ab}(\xi^1, \dots \xi^d)$ are some functions of the coordinates. Note that we can take $g_{ab}(\xi^1, \dots \xi^d) = g_{ba}(\xi^1, \dots \xi^d)$, because the two indices appear symmetrically in the overall sum. This distance measure (11) is called the metric. Thus for the Schwarzschild metric the metric coefficients are

$$g_{11} \equiv g_{tt} = 1 - \frac{2GM}{r}, \quad g_{22} \equiv g_{rr} = -\frac{1}{\frac{1-2GM}{r}}, \quad g_{33} \equiv g_{\theta\theta} = -r^2, \quad g_{44} \equiv g_{\phi\phi} = -r^2 \sin^2\theta \tag{12}$$

with all other coefficients zero.

3 Geodesic motion in the Schwarzschild metric

The motion of particles in general relativity is given by a principle as simple as the one in special relativity: we just extremize the path length from the initial position to the final position. Let us again parametrize the path by a variable τ, so that the functions to be determined are $\xi^a(\tau), a = 1, \ldots d$. Thus the action is

$$S = -m \int_i^f ds = -m \int_i^f \sqrt{\sum_{a=1}^{d} \sum_{b=1}^{d} g_{ab}(\xi^1 \ldots \xi^d) \frac{d\xi^a(\tau)}{d\tau} \frac{d\xi^b(\tau)}{d\tau}} \; d\tau \tag{13}$$

This is now just a general variational problem, with one difference. The parameter τ was arbitrary, and after we have found the equations of motion from the variational principle, we will still have the right to choose it is some particular way if that should prove more convenient.

Let us now specialize to geodesic motion in the Schwarzschild metric. Just as in the Newtonian problem, we can restrict motion to the equatorial plane $\theta = \frac{\pi}{2}$, since reflection symmetry across this plane will not allow the particle to move either above or below this plane if it starts out with positions and velocities in this plane. Thus we will only have the variables $t(\tau), r(\tau), \phi(\tau)$, and the metric coefficients will be taken from (12).

We see that the metric coefficients depend on r, but not on t, θ. Thus t, θ are cyclic coordinates, and there are two conserved quantities

$$p_t = \frac{\partial L}{\partial(\frac{dt}{d\tau})} = \frac{m \sum_c g_{tc} \frac{d\xi^c}{d\tau}}{\sqrt{\sum_{a,b} g_{ab} \frac{d\xi^a}{d\tau} \frac{d\xi^b}{d\tau}}} \tag{14}$$

$$p_\theta = \frac{\partial L}{\partial(\frac{d\theta}{d\tau})} = \frac{m \sum_c g_{\theta c} \frac{d\xi^c}{d\tau}}{\sqrt{\sum_{a,b} g_{ab} \frac{d\xi^a}{d\tau} \frac{d\xi^b}{d\tau}}} \tag{15}$$

At this point we see that there is a good way to choose the arbitrary parameter τ. We can choose it to measure distance ds along the worldline. Thus we will have

$$d\tau^2 = ds^2 = \sum_{ab} g_{ab} d\xi^a d\xi^b \tag{16}$$

This implies that

$$\sum_{a,b} g_{ab} \frac{d\xi^a}{d\tau} \frac{d\xi^b}{d\tau} = 1 \tag{17}$$

which simplifies the expressions for the conserved quantities. For the Schwarzschild metric the above condition becomes

$$(1 - \frac{2GM}{r})(\frac{dt}{d\tau})^2 - \frac{1}{(1 - \frac{2GM}{r})}(\frac{dr}{d\tau})^2 - r^2(\frac{d\theta}{d\tau})^2 = 1 \tag{18}$$

and the conserved quantities are

$$p_t = m(1 - \frac{2GM}{r})\frac{dt}{d\tau} \tag{19}$$

$$p_\theta = -mr^2\frac{d\theta}{d\tau} \tag{20}$$

We thus have

$$\frac{dt}{d\tau} = \frac{p_t}{m(1-\frac{2GM}{r})}, \qquad \frac{d\theta}{d\tau} = -\frac{p_\theta}{mr^2} \tag{21}$$

and the condition (18) becomes

$$\frac{p_t^2}{m^2(1-\frac{2GM}{r})} - \frac{1}{(1-\frac{2GM}{r})}(\frac{dr}{d\tau})^2 - \frac{p_\theta^2}{m^2r^2} = 1 \tag{22}$$

which we rewrite as

$$\frac{1}{2}(\frac{dr}{d\tau})^2 + \Big[\frac{p_\theta^2(1-\frac{2GM}{r})}{2m^2r^2} - \frac{GM}{r}\Big] = \frac{1}{2}(\frac{p_t^2}{m^2} - 1) \tag{23}$$

We can think of this as the energy balance equation for a particle of unit mass moving in the 1-dimensional effective potential

$$V_{eff} = \frac{p_\theta^2(1-\frac{2GM}{r})}{2m^2r^2} - \frac{GM}{r} \tag{24}$$

with effective total energy

$$E_{eff} = \frac{1}{2}(\frac{p_t^2}{m^2} - 1) \tag{25}$$