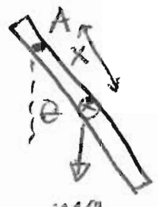
Morin 8.43 Consider a pendulum consisting of a uniform stick (m, L). Where along the stick should one position the pivot if one wants to maximize the oscillation frequency?



Defining the angle θ with respect to vertical, and choosing the origin at the pivot (A) to avoid any $\vec{\tau}$ from the pivot force, the $\vec{\tau}$ eqn reads:

$$\tau_A = -mgx\sin\theta = \frac{d}{dt}L_A = +I_A\ddot{\theta}$$

when θ increases, $\vec{L} \parallel \hat{z}$

where the parallel axis thm says $I_A = mx^2 + I_0$ (and I_0 for our stick is $\frac{1}{12}mL^2$)

(Alternatively, start with the expression $L_A = (\vec{R}_A \times \vec{P})_z + I_0\dot{\theta}$ and note that for this pivoted motion, $\vec{P} = m x \dot{\theta}\hat{\theta}$ so the $\vec{R}_A \times \vec{P} = (x\hat{r}) \times (m x \dot{\theta}\hat{\theta})$ term is the parallel axis term in L_A.) So $\ddot{\theta} = \frac{-mgx}{mx^2+I_0}\sin\theta = -\omega_0^2\sin\theta$

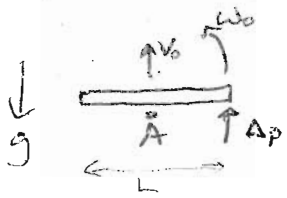
$$\omega_0^2 = \frac{gx}{x^2+I_0/m}$$

To maximize the frequency we can maximize ω_0^2 (equivalently: $\frac{d}{dx}(\omega_0^2) = 0 \Rightarrow$ either $\omega_0 = 0$ or $\omega_0' = 0$)

$$0 \stackrel{!}{=} (\omega_0^2)' = \frac{g}{x^2+I_0/m} - \frac{gx(2x)}{(x^2+I_0/m)^2} \Rightarrow x^2 + \frac{I_0}{m} = 2x^2 \Rightarrow x = \sqrt{\frac{I_0}{m}}$$

For a stick, then choose $x = L/\sqrt{12}$ from the center.

Morin 8.72 A free stick starts at rest as it is struck by a blow at the end. It rises up, spins around and returns to the origin height just as it is horizontal again. How high did the CM go if the stick flipped n half-revolutions?



Choosing an origin in the middle of the stick and calling the linear and angular velocities right after the blow v_0 & ω_0, we have:

$$\Delta p = mv_0 \quad \text{and} \quad \Delta L_A = \left(\frac{L}{2}\right)\Delta p = I_0\omega_0$$

no $R \times P$ term for this choice of origin

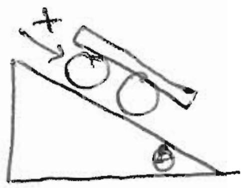
The time for the CM to rise and fall is $T = 2v_0/g$, during which time we want to rotate by an angle $\omega_0 T = n\pi$. And since $\omega_0 = \frac{mL}{2I_0}v_0$ we learn

$$n\pi = \left(\frac{mL}{2I_0}v_0\right)\left(\frac{2v_0}{g}\right) \quad \text{and the CM rises to} \quad h = \frac{v_0^2}{2g} = n\pi\frac{I_0}{2mL} = \frac{n\pi}{24}L$$

Note that the height doesn't depend on m or g, as we could see from the start, using chapter 1 unit analysis ideas. And if we chose an origin at the end of the stick, no angular impulse is given, so we'd write $0 = L_B = -\frac{mv_0L}{2} + I_0\omega_0$ which leads us back to the same result.

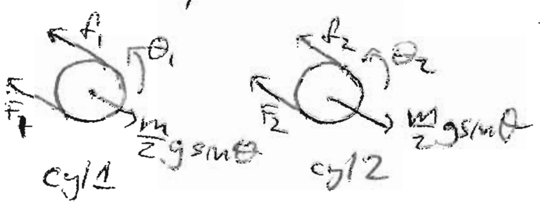
Morin 8.46

A board rides on top of two cylinders as they all careen down a hill. Find the acceleration of the board, assuming no slipping at the various points of contact. For simplicity take $M_{cylinder} = m/2$, where m is the board's mass. Use force & torque!



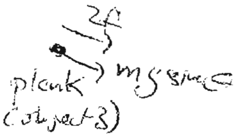
First let's count equations & unknowns, making the counting as excessive as possible. There are 3 objects moving in 2D, so that is 6 x-y type coordinates. Since they can each spin we get 3 more angular variables. Then at each point of contact there is a 2D force (which we decompose as usual into "normal" & "friction"), so that is 8 unknown forces plus 9 coordinates, or 17 equations we need to find. They are: 6 from $\vec{F}=m\vec{a}$, 3 from $\frac{d}{dt}\vec{L}=\vec{\tau}$ for 3 bodies, 4 non-slip constraints at the points of contact, 3 constraints saying nothing moves in the direction perpendicular to the plane, and finally a constraint which says the board doesn't rotate.

But... this is way excessive since we are only interested in the x-acceleration, and in that direction the forces on the two cylinders must be identical:



Linear $\vec{F}=m\vec{a}$ says the sums must agree, while torque says the differences $f_1 - F_1 = f_2 - F_2$, so $f_1 = f_2$ and $F_1 = F_2$

For the cylinders we define $\ddot{x} = \ddot{x}_1 = \ddot{x}_2 = \left[-f - F + \frac{m}{2} g \sin\theta\right]/(m/2)$



while for the board we'll use the symbol $\ddot{y} \equiv \ddot{x}_3 = \left[2f + mg\sin\theta\right]/m$

For the angular variables (defined counterclockwise, just to be definite) torque needs:
(Choosing an origin at the center of the cylinder) $\frac{d}{dt}\left[I_0 \dot{\theta}_1\right] = f_1 R - F_1 R$

But the rolling w/o slipping at the bottom constraint needs: $\dot{x}_1 + \dot{\theta}_1 R \overset{!}{=} 0 \Rightarrow \ddot{\theta}_1 = -\ddot{x}_1/R$

And $I_0 = \frac{1}{2}\left(\frac{m}{2}\right)R^2$, so $-\frac{1}{4} m R^2 \ddot{x}/R = fR - FR \Rightarrow \frac{1}{4} m \ddot{x} = F - f$

Lastly, the no slip at the top needs: $\dot{x} - R\dot{\theta} \overset{!}{=} \dot{y} \Rightarrow \ddot{y} = 2\ddot{x}$ so in the end we get this system for the 3 unknowns $(\ddot{x}, f, F)$:

$$\frac{m}{2}\ddot{x} = -f - F + \frac{m}{2} g \sin\theta \qquad \frac{m}{4}\ddot{x} = F - f \qquad m 2\ddot{x} = 2f + mg\sin\theta$$

$$\Rightarrow \frac{m}{2}\ddot{x} = -f - \underbrace{\left[\frac{m}{4}\ddot{x} + f\right]}_{F} + \frac{m}{2} g\sin\theta = -\frac{m\ddot{x}}{4} - \underbrace{2\left[m\ddot{x} - \frac{mg\sin\theta}{2}\right]}_{2f} + \frac{m}{2} g\sin\theta$$

$$\Rightarrow m\ddot{x}\left[\frac{1}{2} + \frac{1}{4} + 2\right] = mg\sin\theta\left(1 + \frac{1}{2}\right) \Rightarrow \ddot{x} = \frac{6}{11} g\sin\theta$$

And the acceleration of the board is $\ddot{y} = 2\ddot{x} = \frac{12}{11} g \sin\theta$.

Note that we never needed to know the parameters m and R to answer the question about acceleration — unit analysis tells us this from the start.

Morin 7.9 A block (m) slides on a frictionless table, tethered by a rope of ever decreasing length. What quantity is conserved, and what is the speed $v(r)$, given $v(\ell)=v_0$?

Remember the first time we did this? Since $\vec{F}=T(-\hat{r})$ we observed that $0=a_\theta=r\ddot{\theta}+2\dot{r}\dot{\theta}$ and for uniform pulling such that $r(t)=r_0-bt$ we then had a differential equation to solve. This time we just observe that $a_\theta=0$ is equivalent to conservation of ℓ: $\frac{d}{dt}(mr^2\dot{\theta})=0$

So we immediately know $mvr=mv_0\ell \Rightarrow v=v_0\ell/r$

Note that energy is not conserved as we reel the mass in — we computed the work done in another homework.

Morin 7.11 Derive Kepler's 3rd for a circular orbit.

Well... $\vec{F}=m\vec{a}$ reads:

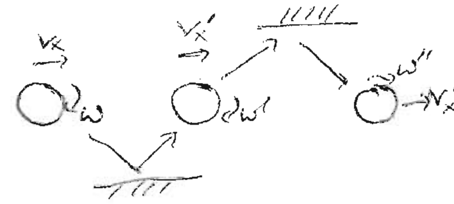
$$-\frac{\alpha}{r^2}=ma_r=m(-r\dot{\theta}^2)$$

$$0=ma_\theta=m(r\ddot{\theta}) \Rightarrow \dot{\theta}=\omega_0 \text{ is constant in time}$$

Since the period is $T=2\pi/\omega_0$, we find $T^2=\left(\frac{2\pi}{\dot{\theta}}\right)^2=\frac{4\pi^2r^3m}{\alpha}=\frac{4\pi^2r^3}{GM}$

in agreement with eqn 7.36 when $a\to r$.

Morin 8.76 Find the condition between V_x and $R\omega$ so that a superball (problem 8.20) launched with these will bounce off the floor, then bounce off the bottom of a table before reversing its path.



In 8.20 we found the relation $\begin{bmatrix}V_x'\\R\omega'\end{bmatrix}=\begin{bmatrix}3&-4\\-10&-3\end{bmatrix}\frac{1}{7}\begin{bmatrix}V_x\\R\omega\end{bmatrix}$ for bouncing off the floor.

Bouncing off the ceiling is exactly the same, except the spin is reversed, i.e. $\begin{bmatrix}V_x''\\-R\omega''\end{bmatrix}=\frac{1}{7}\begin{bmatrix}3&-4\\-10&-3\end{bmatrix}\begin{bmatrix}V_x'\\-R\omega'\end{bmatrix}$

But to reverse path, we are requiring that $V_x''=-V_x'$ and $\omega''=-\omega'$,

That's: $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}-V_x'\\+R\omega'\end{bmatrix}=\begin{bmatrix}3&-4\\-10&-3\end{bmatrix}\frac{1}{7}\begin{bmatrix}V_x'\\-R\omega'\end{bmatrix}$ or: $\left(\begin{bmatrix}0\\0\end{bmatrix}=\frac{1}{7}\begin{bmatrix}7+3&-4\\-10&7-3\end{bmatrix}\begin{bmatrix}V_x'\\-R\omega'\end{bmatrix}\right)$

which says V_x' & $R\omega'$ obey the relation $10V_x'+4R\omega'=0$

And to get V_x' & ω' in this ratio we need the initial parameters to look like

$$\begin{bmatrix}V_x\\R\omega\end{bmatrix}\propto\begin{bmatrix}3&-4\\-10&-3\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}26\\-5\end{bmatrix}$$

In other words the constraint is: $R\omega=\frac{5}{26}V_x$, i.e. light topspin.