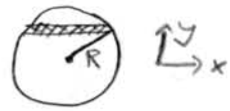
Physics 261 Hwk #9

BTM 3.2.2 asks for the area of a circle to be computed as:

$$A = \int_{-R}^{R} dy \int_{-\sqrt{R^2-y^2}}^{+\sqrt{R^2-y^2}} dx = \int_{-R}^{R} dy \, 2\sqrt{R^2-y^2}$$



Change variables: $y \equiv R\sin\theta \implies dy = R\cos\theta \, d\theta$

$$\implies A = \int_{-\pi/2}^{\pi/2} R\cos\theta \, 2\sqrt{R^2 - R^2\sin^2\theta} = 2R^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta = 2R^2 \pi/2 \quad \checkmark$$

where we use the result of hwk problem BTM 2.1.4 to find $\int \cos^2$

BTM 3.2.4 In spherical coordinates $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$ so the volume of a sphere is found as:

$$V = \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi = \left(\tfrac{1}{3}R^3\right)(2)(2\pi) = \tfrac{4}{3}\pi R^3 \quad \checkmark$$

BTM 3.2.5 To find I for a disk, use polar elements of size $r\,dr\,d\theta$ with mass $dm = \rho \, r\,dr\,d\theta$. For the uniform case, $\rho = M/\pi R^2$ so...

$$I = \int dm \, r^2 = \int_0^R dr \int_0^{2\pi} d\theta \, \rho r^3 = \left(\tfrac{1}{4}R^4\right)(2\pi)\rho = \tfrac{1}{2}MR^2$$

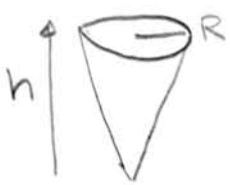
When instead $\rho(r) = \alpha r$ (i.e. linearly increasing density) we expect a larger I (for the same M) since mass has been scooped out of the middle and placed at larger radius. To check, we first determine our α:

$$M \stackrel{!}{=} \int dm = \int \rho \, dA = \int_0^R dr \int_0^{2\pi} r\,dr\,d\theta \, \alpha r = 2\pi\alpha\left(\tfrac{1}{3}R^3\right) \implies \alpha = \frac{3M}{2\pi R^3}$$

Then we find

$$I = \int r^2 dm = \int_0^R dr \, 2\pi\alpha r^4 = 2\pi\alpha R^5/5 = \tfrac{3}{5}MR^2 \quad \left(> \tfrac{1}{2}MR^2\right)$$

BTM 3.2.6



In cylindrical coordinates (r, θ, z) the volume element is $dV = r\,d\theta\,dr\,dz$. As we integrate over $z \in (0, h)$ at a given height the radius is $r(z) = \frac{R}{h}z$. Thus...

$$V = \int dV = \int_0^h dz \int_0^{\frac{R}{h}z} dr \int_0^{2\pi} r\,d\theta = \int_0^h dz \, 2\pi \tfrac{1}{2}\left(\tfrac{R}{h}z\right)^2 = \tfrac{1}{3}\pi R^2 h$$

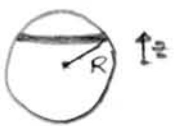
Morin 8.32 Take a cone (M, h, R) and slice it in to disks of thickness dz and radius $r = \frac{R}{h} z$. Since we know the volume of the cone, the mass density is $\rho = M / \frac{1}{3}\pi R^2 h$ and

$$dm = \rho\, dV = \rho \pi r^2 dz = 3M\, z^2 dz / h^3$$

Such a disk has $dI = \frac{1}{2} dm\, r^2 = \frac{3}{2} M R^2 \frac{z^4 dz}{h^5}$

Then adding these up: $I = \int_0^h dI = \frac{3}{10} M R^2$

Morin 8.33 Next slice a sphere (M, R) into a stack of pancakes of thickness dz and radius $r(z) = \sqrt{R^2 - z^2}$. Since $\rho = M / \frac{4}{3}\pi R^3$, these disks have $dm = \rho \pi r^2 dz = \rho \pi (R^2 - z^2) dz$



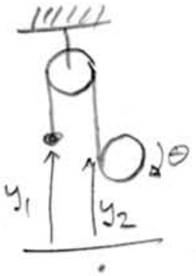
and $dI = \frac{1}{2} dm\, r^2 = \frac{\pi}{2} \rho (R^2 - z^2)^2 dz$

Adding up all the disks between $z = -R$ and $z = +R$:

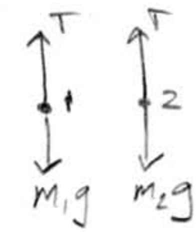
$$I = \int dI = \frac{\pi\rho}{2} \int_{-R}^{R} \left[R^4 - 2R^2 z^2 + z^4\right] dz = \pi \rho R^5 \left[1 - 2 \cdot \frac{1}{3} + \frac{1}{5}\right]$$

$$= \frac{3}{4} M R^2 \left[\frac{8}{15}\right] = \frac{2}{5} M R^2$$

Morin 8.27 A mass (m) and a spool (also of mass m, and radius R) are draped over a standard issue massless frictionless pulley, using massless inextensible rope. Find the accelerations.



Energy solution. Per instructions, we first note that $\ddot{y}_1 = \ddot{y}_2$, because when $m_1 = m_2$ the force diagrams for 1 & 2 are exactly the same:



Then we suppose that we release the masses at rest and wait for them each to descend a distance h, thus releasing $2mgh$ of potential energy, which must show up as kinetic:

$$2mgh = \underbrace{\tfrac{1}{2}mv^2}_{\text{mass 1}} + \underbrace{\left[\tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2\right]}_{\text{spool}}$$

The tricky part is getting the constraint right — observe that for every meter m_1 & m_2 descend, we need 2 meters of rope to come off the spool. Thus $\omega \equiv \dot{\theta} = 2V/R$

$$\Rightarrow 2mgh = mv^2 + \tfrac{1}{2}\Big(\underbrace{\tfrac{1}{2}mR^2}_{I \text{ for disk}}\Big)\left(\frac{2V}{R}\right)^2 \Rightarrow gh = v^2$$

On the other hand, for uniform acceleration a it takes a time $t = V/a$ to accelerate from 0 to v, during which time we fall a distance $h = \frac{1}{2}at^2 = \frac{v^2}{2a} \Rightarrow \boxed{a = \frac{v^2}{2h} = \frac{1}{2}g}$

Morin 8.45 Force/torque method. From $\vec{F} = m\vec{a}$ we learn:

Choosing origin A directly below the CM we both kill the $\vec{R} \times \vec{P}$ term in $\vec{L}$, and the mg contribution to the torque.

$$\left.\begin{aligned} m\ddot{y}_1 &= T - mg \\ m\ddot{y}_2 &= T - mg \end{aligned}\right\} \Rightarrow \ddot{y}_1 = \ddot{y}_2 = \ddot{y}$$

$\frac{d}{dt}L = \tau$ then reads: $\frac{d}{dt}(-I\omega) = \tau = -TR$

into the page

The final ingredient is the rope length constraint:

$$\underbrace{l_0}_{\text{length of rope}} = \underbrace{(h - y_1)}_{\text{left piece}} + \underbrace{(h - y_2)}_{\text{right piece}} + \underbrace{(L - R\theta)}_{\text{amount on spool}} \Rightarrow 0 = \ddot{y}_1 + \ddot{y}_2 + R\ddot{\theta}$$

note that as θ increases there is less rope on the spool

So now we have 4 relations among 4 unknowns $(\ddot{y}_1, \ddot{y}_2, \ddot{\theta}, T)$

Solving: Eliminate T with $T = m(g + \ddot{y}) \Rightarrow \ddot{\theta} = \frac{TR}{I} = \frac{m(g+\ddot{y})R}{\frac{1}{2}mR^2} = \frac{2(g+\ddot{y})}{R}$

Then the constraint reads: $0 = 2\ddot{y} + R\ddot{\theta} = 2\ddot{y} + 2(g + \ddot{y})$

$$\Rightarrow \ddot{y} = -\tfrac{1}{2}g \quad \text{just as above}$$

Note that the energy method is simplified considerably by $m_1 = m_2 = m$, while the force/torque method will work the same way for $m_1 \neq m_2$

Morin 8.38 A can rolls down an inclined plane. If the coefficient of friction is μ, what is the steepest angle θ we can have w/o slipping?

Start with the linear force diagram:

$\Rightarrow m\dot{v} = mg\sin\theta - f$

$N = mg\cos\theta$

Then choosing origin A in line with the CM motion, the torque relation is:

N & $mg\cos\theta$ torques cancel

$$\tau = -fR = \frac{d}{dt}L = -I\dot{\omega}$$

The rolling w/o slipping constraint is: $v = \omega R \Rightarrow \dot{v} = \dot{\omega}R$

And the "steepest" condition implies $f = \mu N$

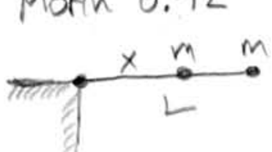
That makes 5 relations among $(\dot{v}, \dot{\omega}, N, f \text{ and } \theta)$. Going after $\dot{\omega}$ and $\dot{v}$:

$$\dot{\omega} = \frac{fR}{I} = \frac{\mu NR}{I} = \frac{\mu mgR}{I}\cos\theta \qquad \dot{v} = mg\sin\theta - f = mg(\sin\theta - \mu\cos\theta)$$

Then relating them: $\dot{\omega}R = \dot{v} \Rightarrow \frac{\mu}{c}\cos\theta = \sin\theta - \mu\cos\theta$ where $c \equiv \frac{I}{mR^2}\left(=\frac{1}{2}\right)$

Thus the critical angle is when $\tan\theta = \mu\left(1+\frac{1}{c}\right) = 3\mu$ for a disk.

Morin 8.42



A massless rod of length L is pivoted at one end. At the other end is a fixed mass m. In addition there is a moveable mass m positioned at x. Holding the rod horizontally, how should we choose x to maximize the angular acceleration?

Well... the torque from the two masses (choosing the origin @ the pivot of course) is:

$$\tau = mgx + mgL = mgL(1+\beta) \text{ where we define the dimensionless } \beta \equiv x/L$$

And given a moment of inertia $I = mx^2 + mL^2 = mL^2(1+\beta^2)$, the angular acceleration is:

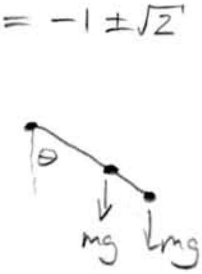
$$\alpha = \frac{\tau}{I} = \frac{g}{L}\frac{1+\beta}{1+\beta^2} \equiv \frac{g}{L}f(\beta) \text{ where } f(\beta) \equiv \frac{1+\beta}{1+\beta^2}$$

Maximizing f: $0 \stackrel{!}{=} f'(\beta) = \frac{1\cdot(1+\beta^2) - (1+\beta)(2\beta)}{(1+\beta)^3} \Rightarrow 0 = +\beta^2 + 2\beta - 1$

$\Rightarrow \beta = -1 \pm \sqrt{2}$

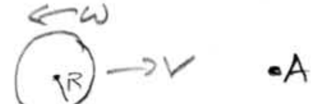
Plainly the root we want is the positive one, i.e. choose $x = (\sqrt{2}-1)L$

Note that the "horizontal" condition is just for convenience — at an arbitrary angle θ, the torque is $\tau = mg\sin\theta\,(x+L)$ and the angular acceleration is again maximized by the same x.



Also note we could ask for the greatest angular speed ω when the rod falls to vertical. In that case the potential energy released is $mgL + mgx$, which is to be equated with $\frac{1}{2}I\omega^2$, so again we choose x to maximize $\frac{x+L}{x^2+L^2}$.

Morin 8.48 A coin is launched with speed v and spin ω : It slides across the floor which has friction μ. How should we choose v & ω so the hoop stops dead a distance d from launch?



Plainly the friction force is $f = \mu N = \mu m g$ which will stop the linear motion in time $t = v/\mu g$ which tells us to choose v such that $d = \frac{1}{2}(\mu g)t^2 = \frac{v^2}{2\mu g}$

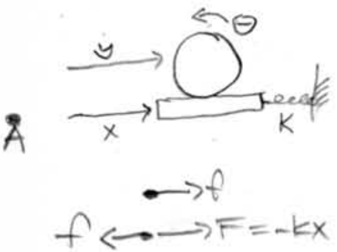
$\Longrightarrow \boxed{v = \sqrt{2\mu g d}}$ To pin down ω we choose our origin A in line with the CM so that $L_0 = I\omega$ and $\tau = -Rf = -R\mu m g$

Over the time t this torque delivers an angular impulse $\tau t = -Rmv$

So if we want the coin to stop spinning at t we should choose

$\boxed{\omega = \frac{Rmv}{I} = 2v/R}$ for the case $I = \frac{1}{2}mR^2$

Morin 8.52 A cylinder (m, R) rolls w/o slipping on top of a board (m) which is attached to a spring (k). Find the frequency of oscillation



F=ma's: $m\ddot{y} = f$ $\quad m\ddot{x} = -f - kx$

nonslip constraint: $\dot{y} + \dot{\theta}R = \dot{x} \Longrightarrow \ddot{y} + \ddot{\theta}R = \ddot{x}$

torque (w/ origin A): $\tau = 0 = \frac{d}{dt}\left[-Rm\dot{y} + I\dot{\theta}\right] = -Rm\ddot{y} + I\ddot{\theta}$

Solving for $(\ddot{x}, \ddot{y}, \ddot{\theta}, f)$ we find $\ddot{x} = -\frac{3}{4}\frac{k}{m}x \Longrightarrow \omega = \sqrt{\frac{3k}{4m}}$

See the Mathematica notebook for an animation.